ON THE UTILITY OF ROBINSON–AMITSUR ULTRAFILTERS

PASHA ZUSMANOVICH

Abstract. Two similar embedding theorems for algebras and groups are presented, basing on a certain old ultrafilter construction. As an application, we outline alternative proofs of some results from the theory of PI algebras, and establish some interesting properties of Tarski's monsters.

Introduction

In 1960s, Robinson and Amitsur established a number of embedding results in Ring Theory which proved to be useful in various structural questions. A typical example: if a prime ring R embeds in the direct product of associative division rings, then R embeds in an associative division ring (see, for example, [E, §6]).

The proof of these results follows the same scheme: basing on the initial data – a ring embedded in the direct product of rings – a certain ultrafilter, which we call a *Robinson–Amitsur ultrafilter*, is constructed. Using this ultrafilter, one passes from the direct product of rings to their ultraproduct, and appeal to Łoś' theorem about elementary equivalence of an algebraic system and its ultraproduct completes the proof.

In this note we, first, extend this argument to the case of arbitrary (nonassociative) algebras (Theorem 1.2), and, second, present a similar one for groups (Theorem 2.1). Coupled with the classical Birkhoff's theorem about varieties of algebraic systems, this gives a simple yet elegant criterion for an algebra or group not to satisfy a nontrivial identity (Corollaries 1.7 and 2.3). The corresponding group result is not entirely new (see comments after Corollary 2.2), but, we believe, its proof is, and the links between apriori unrelated concepts, ideas and results is the only novelty, if any, of this note.

As an application, we outline alternative, "by abstract nonsense", proofs of some particular cases of the well-known results from the theory of PI algebras (§3), of results about algebras having the same identities (§4), and establish that Tarski's monsters without identities have infinite (relative) girth (§5).

The narrative is occasionally interspersed with questions and speculations.

Notation and conventions

In what follows, by just "algebra" or "ring", we mean an arbitrary, not necessary associative, or Lie, or satisfying any other distinguished identities, algebra or ring, unless it is stated otherwise.

Date: last revised January 23, 2010.

A previous version of this manuscript is available as arXiv:0911.5414.

Recall the construction of the ultraproduct. Let $\{A_i\}_{i\in\mathbb{I}}$ be a set of algebras, and $\mathscr{D}$ is an ultrafilter on the set $\mathbb{I}$. Then

$$\mathcal{I}\Big(\prod_{i\in\mathbb{I}} A_i, \mathscr{D}\Big) = \{f \in \prod_{i\in\mathbb{I}} A_i \mid \{i \in \mathbb{I} \mid f(i) = 0\} \in \mathscr{D}\}$$

is an ideal of the direct product $\prod_{i\in\mathbb{I}} A_i$, and the quotient by this ideal is called *ultraproduct* of the set $\{A_i\}_{i\in\mathbb{I}}$ with respect to ultrafilter $\mathscr{D}$, and is denoted as $\prod_{\mathscr{D}} A_i$. In the particular case where all A_i's are isomorphic to the same algebra A, their ultraproduct is called *ultrapower* of A and is denoted as $A^{\mathscr{D}}$.

The construction for groups is similar. We refer to [E] and [BS] for all necessary notions and results related to ultrafilters and ultraproducts.

If A is an algebraic system (algebra or group), $Var(A)$ denotes a variety generated by A.

1. Algebras

Let $\mathfrak{V}$ be a variety of algebras, and $\mathcal{F}(X)$ is a free algebra in this variety generated by a set X. By *words* we mean elements of $\mathcal{F}(X)$ for some X. The standard grading on $\mathcal{F}(X)$ is defined by length of words. By *2-nontrivial words* we mean words degrees of all whose homogeneous components in the first two indeterminates are non zero. For example,

$$xy, \quad (xy)x - (xy)(xz), \quad (zy)x + 2(tx)y + x(yz)(tx)$$

are 2-nontrivial words in x, y, z, t (in that order), while

$$x, \quad xz, \quad (xy)x + xz, \quad (yz)(yt) + (xy)z$$

are not.

Lemma 1.1. *For an algebra $A \in \mathfrak{V}$, the following is equivalent:*

(i) *For any two nonzero ideals $I, J \subseteq A$, $IJ \neq 0$.*
(ii) *For any two nonzero elements $x, y \in A$, there is a 2-nontrivial word $w(\xi_1, \dots, \xi_n)$, $n \geq 2$ and elements $x_1, \dots, x_{n-2} \in A$ such that $w(x, y, x_1, \dots, x_{n-2}) \neq 0$.*

Proof. (i) $\Rightarrow$ (ii). Suppose there are nonzero $x, y \in A$ such that for any 2-nontrivial word $w(\xi_1, \dots, \xi_n)$ and any $x_1, \dots, x_{n-2} \in A$, $w(x, y, x_1, \dots, x_{n-2}) = 0$. Let I and J be ideals of A generated by x and y respectively. Clearly $IJ = 0$, a contradiction.

(ii) $\Rightarrow$ (i). Suppose I, J are two nonzero ideals of A. Taking $x \in I$ and $y \in J$, we have $w(x, y, x_1, \dots, x_{n-2}) \neq 0$ for some 2-nontrivial word w and elements x_i of A. But clearly $w(x, y, x_1, \dots, x_{n-2}) \in IJ$, hence $IJ \neq 0$. □

An algebra $A \in \mathfrak{V}$ satisfying the equivalent conditions of Lemma 1.1, is called $\mathfrak{V}$-*prime*. When $\mathfrak{V}$ is a variety of all associative algebras, this notion coincides with the classical notion of a prime associative algebra.

Clearly, if $\mathfrak{W}$ is another variety and $\mathfrak{V} \subseteq \mathfrak{W}$, an algebra $A \in \mathfrak{V}$ is $\mathfrak{V}$-prime if and only if it is $\mathfrak{W}$-prime, so we can speak about just *prime algebras* (which are prime in the variety of all algebras).

Theorem 1.2 (Robinson–Amitsur for algebras)**.** *Let $\{B_i\}_{i\in\mathbb{I}}$ be a set of algebras. If a prime algebra A embeds in the direct product $\prod_{i\in\mathbb{I}} B_i$, then A embeds in an ultraproduct $\prod_{\mathscr{D}} B_i$.*

Proof. Define
$$\mathscr{S} = \{\{i \in \mathbb{I} \mid f(i) \neq 0\} \mid f \in A, f \neq 0\}.$$
Let us verify that intersection of any two elements of $\mathscr{S}$ contains an element of $\mathscr{S}$. Let $M, N \in \mathscr{S}$, say, $M = \{i \in \mathbb{I} \mid f(i) \neq 0\}$ and $N = \{i \in \mathbb{I} \mid g(i) \neq 0\}$ for some nonzero $f, g \in A$. Since A is prime, $w(f, g, h_1, \dots, h_{n-1}) \neq 0$ for some 2-nontrivial word w and $h_1, \dots, h_{n-2} \in A$, and, consequently,
$$M \cap N \supset \{i \in \mathbb{I} \mid w(f, g, h_1, \dots, h_{n-2})(i) \neq 0\} \in \mathscr{S}.$$

Thus $\mathscr{S}$ satisfies the finite intersection property and contained in a certain ultrafilter $\mathscr{D}$ on $\mathbb{I}$. Factoring the embedding of algebras $A \hookrightarrow \prod_{i\in\mathbb{I}} B_i$ through the ideal $\mathcal{I}(\prod_{i\in\mathbb{I}} B_i, \mathscr{D})$, we get an embedding of algebras
$$A/\Big(A \cap \mathcal{I}\Big(\prod_{i\in\mathbb{I}} B_i, \mathscr{D}\Big)\Big) \hookrightarrow \Big(\prod_{i\in\mathbb{I}} B_i\Big)/\mathcal{I}\Big(\prod_{i\in\mathbb{I}} B_i, \mathscr{D}\Big) = \prod_{\mathscr{D}} B_i.$$
Let $f \in A \cap \mathcal{I}\Big(\prod_{i\in\mathbb{I}} B_i, \mathscr{D}\Big)$. Then $\{i \in \mathbb{I} \mid f(i) = 0\} \in \mathscr{D}$, and, since $\mathscr{D}$ is ultrafilter, $\{i \in \mathbb{I} \mid f(i) \neq 0\} \notin \mathscr{D}$, and hence $\{i \in \mathbb{I} \mid f(i) \neq 0\} \notin \mathscr{S}$. From $f \in A$ and the definition of $\mathscr{S}$ follows that $f = 0$. This shows that $A \cap \mathcal{I}\Big(\prod_{i\in\mathbb{I}} B_i, \mathscr{D}\Big) = 0$. ☐

Question 1.1. *Is the converse of Theorem 1.2 true, at least in the class of associative algebras? That is, suppose that for an algebra A the following holds: for any set of algebras $\{B_i\}_{i\in\mathbb{I}}$, if A embeds in the direct product $\prod_{i\in\mathbb{I}} B_i$, then A embeds in an ultraproduct $\prod_{\mathscr{D}} B_i$. Does this imply that A is prime? that A satisfies any other natural structural condition?*

The ultraproduct construction used in the proof Theorem 1.2 mimics the old one, used by Robinson and Amitsur in Ring Theory, mentioned in the introduction.

Embedding stated in Theorem 1.2 is embedding of K-algebras defined over the same base field K over which all the algebras in question are defined. Of course, the ultraproduct $\prod_{\mathscr{D}} B_i$ is defined also over the ultrapower field $K^{\mathscr{D}}$, so we have an embedding of $K^{\mathscr{D}}A$ in $\prod_{\mathscr{D}} B_i$ as $K^{\mathscr{D}}$-algebras. Due to the universal property of the tensor product, there is a surjection of $K^{\mathscr{D}}$-algebras

$$A \otimes_K K^{\mathscr{D}} \to K^{\mathscr{D}}A, \qquad (1)$$

but this surjection is, in general, not a bijection.

One may eliminate hurdles related to the ground field altogether by considering Theorem 1.2 as a statement about embeddings of rings. In this way we get a nonassociative analog of one of the classical embedding results in Ring Theory mentioned in the introduction:

Corollary 1.3. *If a prime ring A embeds in the direct product of division rings, then A embeds in a division ring.*

Proof. By Theorem 1.2, A embeds in an ultraproduct of division rings. As the property to be a division ring is the first-order property, by Łoś' theorem the ultraproduct of division rings is a division ring, whence the conclusion. ☐

Another application is to varieties of algebras:

Corollary 1.4. *Let B be an algebra, and A is a relatively free algebra in $Var(B)$. If A is prime, then it embeds in an ultrapower of B.*

Proof. According to Birkhoff's theorem, $A = \mathcal{F}/\mathcal{I}$ for an ideal $\mathcal{I}$ of an algebra $\mathcal{F}$, and $\mathcal{F}$ is a subalgebra in a direct power of B. Because of the universal property of A, the short exact sequence $0 \to \mathcal{I} \to \mathcal{F} \to A \to 0$ splits, i.e., A embeds in $\mathcal{F}$, and hence in a direct power of B. Then apply Theorem 1.2. □

The condition of Corollary 1.4 – that a relatively free algebra in a variety is prime – is fulfilled for the major varieties of algebras considered in the literature: all algebras, associative, and Lie (on the other hand, it is not fulfilled for Jordan and alternative algebras). Indeed, for free algebras and free associative algebras this is obvious, as they do not have zero divisors. Now, let I, J be two ideals of a free Lie algebra. By the Shirshov–Witt theorem about freeness of subalgebras of a free Lie algebra, neither of I, J could be one-dimensional, and hence we may choose two linearly independent elements $x \in I$ and $y \in J$. Their commutator is nonzero, as otherwise they form a 2-dimensional abelian subalgebra, what again contradicts the Shirshov–Witt theorem. Hence $[I, J] \neq 0$.

It fulfilled also for any variety generated by a prime algebra itself:

Lemma 1.5. *Let A be a prime algebra over an infinite field. Then the countably-generated relatively free algebra in $Var(A)$ is prime.*

Proof. Let $\mathcal{F}(X)$ be the relatively free algebra in $Var(A)$ freely generated by an infinite set $X = \{x_1, x_2, \dots\}$. Suppose that there are nonzero elements $u(x_1, \dots, x_n)$, $v(x_1, \dots, x_n) \in \mathcal{F}(X)$ such that

$$w(u, v, u_1, \dots, u_m) = 0 \tag{2}$$

for any 2-nontrivial word w and $u_1, \dots, u_m \in \mathcal{F}(X)$. As any relation between free generators of $\mathcal{F}(X)$ is an identity in A, the equalities (2) are identities in A. Taking $u_1 = x_{n+1}, u_2 = x_{n+2}, \dots$, we get that

$$w(u(x_1, \dots, x_n), v(x_1, \dots, x_n), x_{n+1}, \dots, x_{n+m}) = 0$$

for any 2-nontrivial word w and any $x_1, \dots, x_n, x_{n+1}, \dots, x_{n+m} \in A$. As A is prime, either $u(x_1, \dots, x_n) = 0$ or $v(x_1, \dots, x_n) = 0$ for any $x_1, \dots, x_n \in A$.

Both sets of n-tuples of elements of A on which u, respectively v, does not vanish, form a nonempty Zariski-open subset in $A \times \dots \times A$ (n times), whence they have a nonzero intersection, a contradiction. □

Another important observation is that for relatively free algebras A in the varieties of all algebras, associative algebras, and Lie algebras, the surjection (1) is a bijection.

Lemma 1.6. *Let A be a subalgebra of an algebra B, both defined over a field K. Suppose B is also defined over a field F containing K, and that as a K-algebra, A does not have commutative subalgebras of dimension > 1. Then $FA \simeq A \otimes_K F$ as F-algebras.*

Proof. The claimed isomorphism follows from the fact that the linear dependence of elements of A over F implies their linear dependence over K. Indeed, consider the linear dependence

$$f_1 a_1 + \dots + f_n a_n = 0, \tag{3}$$

where $f_i \in F$, $a_i \in A$, and let us prove by induction on n that $a_1, \dots, a_n$ are linear dependent over K. For $n = 1$ this is trivial. Taking commutator with a_n of the both sides of (3), we get

$$f_1[a_1, a_n] + \cdots + f_{n-1}[a_{n-1}, a_n] = 0.$$

(we use the usual notation for the commutator: $[a, b] = ab - ba$). By induction assumption, there are $k_1, \dots, k_{n-1} \in K$ such that

$$k_1[a_1, a_n] + \cdots + k_{n-1}[a_{n-1}, a_n] = [k_1 a_1 + \cdots + k_{n-1} a_{n-1}, a_n] = 0.$$

But since A does not have commutative subalgebras of dimension > 1,

$$k(k_1 a_1 + \cdots + k_{n-1} a_{n-1}) + \ell a_n = 0$$

for some $k, \ell \in K$. □

Obviously, the condition of this Lemma is satisfied when A is a free algebra or a free Lie algebra in > 1 free generators (due to Shirshov–Witt theorem). A similar, but just a little bit more involved argument (see, for example, proof of Lemma 1 in [MM]) shows that the same conclusion holds when A is a free associative algebra in > 1 free generators.

All these observations lead to

Corollary 1.7 (Criterion for absence of non-trivial identities of algebras)**.** *For an algebra B belonging to one of the following variety of algebras: all algebras, associative, or Lie, the following is equivalent:*

(i) *B does not satisfy a nontrivial identity;*
(ii) *a free algebra embeds in an ultrapower of B;*
(iii) *a free algebra embeds in an algebra elementary equivalent to B.*

Proof. (i) ⇒ (ii) follows from Corollary 1.4. Note that due to Lemma 1.6 (applied to the case $F = K^{\mathscr{D}}$, where K is the ground field, and $\mathscr{D}$ is the appropriate ultrafilter) and remarks after it, the embedding of Corollary 1.4 could be considered as an embedding of $K^{\mathscr{D}}$-algebras.

(ii) ⇒ (iii) By Łoś' theorem, the pairs (B, K) and $(B^{\mathscr{D}}, K^{\mathscr{D}})$ are elementary equivalent as models of the two-sorted theory (algebra over a field, field).

(iii) ⇒ (i) An algebra elementary equivalent to B does not satisfy a nontrivial identity. Since the latter is the first-order property, B does not satisfy a nontrivial identity too. □

Remarks. (1) Of course, the equivalence of conditions (ii) and (iii) above also follows from Keisler's ultrapower theorem (see, for example, [BS, Chapter 7, Corollary 2.7]).

(2) Recall the well-known fact from the model theory: an algebra A embeds in an ultrapower of an algebra B if and only if $Th_\forall(A) \subseteq Th_\forall(B)$, where $Th_\forall$ denotes the universal theory of an algebra (see, for example, [BS, Chapter 9, Lemma 3.8]). This allows to add another equivalent condition to Corollary 1.7: a universal theory of a free algebra is contained in the universal theory of B.

Let us provide some negative examples showing that conclusions of some statements of this section do not always hold.

The conclusion of Lemma 1.6 does not hold for polynomial algebras in > 1 variables (i.e., for free associative commutative algebras). Indeed, let $F = K((x))$ (the field of rational function in 1 variable) and $A = K[x, y]$ (the algebra of polynomial in 2 variables). Then

$$FA = K((x))[y] \simeq K[y] \otimes_K K((x)),$$

but $A \otimes_K F = K[x,y] \otimes_K K((x))$.

In some other situations, the surjection (1) also can be very far from being a bijection. For example, consider the relatively free countably-generated algebra $\mathcal{F}$ in the variety $Var(A)$ generated by a finite-dimensional prime algebra over an infinite filed K. By Lemma 1.5, $\mathcal{F}$ is prime, and by Corollary 1.4, $\mathcal{F}$ embeds, as a K-algebra, in an ultrapower $A^{\mathscr{D}} \simeq A \otimes_K K^{\mathscr{D}}$ (see Corollary 3.3 below). Hence $K^{\mathscr{D}}\mathcal{F}$ is a finite-dimensional $K^{\mathscr{D}}$-algebra. On the other hand, since $\mathcal{F}$ is residually nilpotent (see, for example, [Ba, §4.2.10] for the case of Lie algebras; the general case is treated identically), the finite-dimensionality of $\mathcal{F}$ implies nilpotency of $\mathcal{F}$, and hence nilpotency of A, a contradiction. Consequently, $\mathcal{F}$ is infinite-dimensional over K, and $\mathcal{F} \otimes_K K^{\mathscr{D}}$ is infinite-dimensional over $K^{\mathscr{D}}$. Essentially the same type of arguments we will use below in §4, when describing an alternative approach to Razmyslov's result about varieties generated by simple finite-dimensional Lie algebras.

2. Groups

For groups, the role of the algebra primeness property which guarantees the existence of Robinson–Amitsur ultrafilter, is played by the group property that every abelian subgroup is cyclic of prime or of infinite order.

Theorem 2.1 (Robinson–Amitsur for groups). *Let $\{F_i\}_{i\in\mathbb{I}}$ be a set of groups and G is a group such that every its abelian subgroup is cyclic either of a prime order, or of infinite order. If G embeds in the direct product $\prod_{i\in\mathbb{I}} F_i$, then G embeds in an ultraproduct $\prod_{\mathscr{D}} F_i$.*

Proof. The proof goes similar to the proof of Theorem 1.2, the only difference being in establishing the finite intersection property of the set

$$\mathscr{S} = \{\{i \in \mathbb{I} \mid f(i) \neq 1\} \mid f \in G, f \neq 1\},$$

which goes as follows.

Let $M, N \in \mathscr{S}$, say, $M = \{i \in \mathbb{I} \mid f(i) \neq 1\}$ and $N = \{i \in \mathbb{I} \mid g(i) \neq 1\}$ for some $f, g \in G$ different from 1. Suppose first that f and g do not commute, i.e. $h = f^{-1}g^{-1}fg$ is different from 1. Obviously, either of $f(i) = 1$ or $g(i) = 1$ implies $h(i) = 1$ for any $i \in \mathbb{I}$. Consequently,

$$M \cap N \supset \{i \in \mathbb{I} \mid h(i) \neq 1\} \in \mathscr{S}. \tag{4}$$

Suppose now that f and g commute. Then they lie either in a cyclic subgroup $\mathbb{Z}_p$, generated by an element $a \in G$ of a prime order p, or in an infinite cyclic subgroup $\mathbb{Z}$, generated by an element $a \in G$ of infinite order. Write $f = a^n$ and $g = a^m$ for some $0 < n, m < p$ in the first case, and $n, m \in \mathbb{Z}\backslash\{0\}$ in the second one, and let $h = a^{nm} = f^m = g^n$. Note that in both cases, due to restriction on order of a, $h \neq 1$. Now again, either of $f(i) = 1$ or $g(i) = 1$ implies $h(i) = 1$ for any $i \in \mathbb{I}$, and (4) holds also in this case. □

The following question about inverse of this Theorem is similar to Question 1.1:

Question 2.1. *Suppose that a group G satisfies the conclusion of Theorem 2.1: that is, for any set $\{F_i\}_{i\in\mathbb{I}}$ of groups, if G embeds in the direct product $\prod_{i\in\mathbb{I}} F_i$, then G embeds in an ultraproduct $\prod_{\mathscr{D}} F_i$. Could G be characterized in terms of some structural properties?*

Corollary 2.2.

(i) *If G is a group which does not satisfy any nontrivial identity except $x^p = 1$ for a certain prime p, and its consequences, then a free Burnside group embeds in an ultrapower of G.*

(ii) *If G is a group which does not satisfy any nontrivial identity, then a free group embeds in an ultrapower of G.*

Proof. Similarly to Corollary 1.4, the proof consists of appealing to Birkhoff's theorem, and remark that the condition on identities of G means that all abelian subgroups of a relatively free group in $Var(G)$ are cyclic either of order p in the case (i), or of infinite order in the case (ii), and hence Theorem 2.1 is applicable. □

This Corollary is probably known to experts – at least statements equivalent to the part (ii) can be found in [Bo, Theorem 1] and [DS, Lemma 6.15]. The proofs there are different and based on the fact that ultraproducts are ω_1-compact.

Corollary 2.3 (Criterion for absence of non-trivial identities of groups)**.** *For a group G the following is equivalent:*

(i) *G does not satisfy a nontrivial identity;*

(ii) *a nonabelian free group embeds in an ultrapower of G;*

(iii) *a nonabelian free group embeds in a group elementary equivalent to G.*

Proof. Note that by the Nielsen–Schreier theorem, every subgroup of a free group is free, hence every its abelian subgroup is infinite cyclic, so a free group satisfies the condition of Theorem 2.1. The rest of the proof is the same as of Corollary 1.7. □

The same remarks as those after Corollary 1.7 apply also in the group case. In particular, the condition (ii) above is equivalent to the condition that the universal theory of a free group is contained into the universal theory of G.

Question 2.2. *Is there a semigroup property such that the corresponding analogs of the results of this section would hold for semigroups?*

3. Application: PI

As an application of this machinery, let us demonstrate how one can handle, in a way different from the traditional approaches, some well-known statements from the theory of associative algebras satisfying polynomial identities (usually called PI by associative algebraists).

The celebrated Regev's "$A \otimes B$ theorem" asserts that the tensor product of two PI algebras A and B is PI (see, for example, [KR, Theorem 5.42]). If we want to prove it using results of §1, we encounter a few difficulties: first, to establish relationship between the ultrapower of the tensor product $(A \otimes B)^{\mathscr{D}}$ and the tensor product of ultrapowers $A^{\mathscr{D}} \otimes B^{\mathscr{D}}$ (perhaps, considering some sort of completed tensor product instead of the usual one may help), and, second, to be able to say something about algebras A and B such that their (possibly completed) tensor product contains a free associative algebra. But at least, in this way we are able to provide an alternative proof of a particular case where one of the tensor factors is finite-dimensional (first established by Procesi and Small in [PS] for even more particular case where one of the tensor factors is a full matrix algebra). This particular case is morally important, as semiprime PI algebras embed in matrix algebras over commutative rings (see, for example, [KR, Remark 1.69]), what essentially reduces the situation to a finite-dimensional one.

Theorem 3.1 ("Baby Regev's $A \otimes B$")**.** *The tensor product of two associative algebras, one of them PI and another one is finite-dimensional, is PI.*

Lemma 3.2. *Let A, B be algebras defined over a field K, and A is finite-dimensional. Then, for any ultrafilter $\mathscr{D}$,*

$$(A \otimes_K B)^{\mathscr{D}} \simeq A \otimes_K B^{\mathscr{D}}$$

(as K-algebras).

The special cases of this assertion were proved many times in literature – for example, in [NN] for the case where A is an associative full matrix algebra, and in [T, Proposition 25] for the case where both A and B are finite-dimensional, and the proof is standard.

Proof. Let $\{a_1, \dots, a_n\}$ be a basis of A. Obviously, for each K-algebra C, each element of $A \otimes_K C$ could be uniquely represented as $\sum_{k=1}^n a_k \otimes c_k$ for some $c_k \in C$. Define a map

$$\varphi : (A \otimes_K B)^{\mathbb{I}} \to A \otimes_K B^{\mathbb{I}}$$

as follows: for $f \in (A \otimes_K B)^{\mathbb{I}}$ write $f(i) = \sum_{k=1}^n a_k \otimes b_{ki}$, $i \in \mathbb{I}$ and define $\varphi(f) \in A \otimes_K B^{\mathbb{I}}$ as $\sum_{k=1}^n a_k \otimes g_k$, where $g_k \in B^{\mathbb{I}}$ is defined as $g_k(i) = b_{ki}$, $i \in \mathbb{I}$. Writing multiplication in A in terms of the basis elements, one can see that φ is an isomorphism of K-algebras.

The ideal $\mathcal{I}((A \otimes_K B)^{\mathbb{I}}, \mathscr{D})$ maps under φ to $A \otimes_K \mathcal{I}(B^{\mathbb{I}}, \mathscr{D})$, so factoring out both sides of the isomorphism φ by the corresponding ideals, we get:

$$\begin{aligned}(A \otimes_K B)^{\mathscr{D}} = (A \otimes_K B)^{\mathbb{I}} / \mathcal{I}((A \otimes_K B)^{\mathbb{I}}, \mathscr{D}) &\simeq (A \otimes_K B^{\mathbb{I}}) / (A \otimes_K \mathcal{I}(B^{\mathbb{I}}, \mathscr{D})) \\ &\simeq A \otimes_K (B^{\mathbb{I}} / \mathcal{I}(B^{\mathbb{I}}, \mathscr{D})) = A \otimes_K B^{\mathscr{D}}.\end{aligned}$$

□

Corollary 3.3 ([T], Proposition 21)**.** *Let A be a finite-dimensional algebra defined over a field K. Then, for any ultrafilter $\mathscr{D}$, $A^{\mathscr{D}} \simeq A \otimes_K K^{\mathscr{D}}$ (as K-algebras).*

Proof. Put $B = K$. □

Proof of Theorem 3.1. Let A be a finite-dimensional associative algebra, and B is PI, defined over a field K. Suppose $A \otimes_K B$ is not PI. Then by Corollary 1.7, some ultrapower $(A \otimes_K B)^{\mathscr{D}}$, considered as a $K^{\mathscr{D}}$-algebra, contains a free finitely-generated associative subalgebra $\mathcal{F}$. By Lemma 3.2,

$$(A \otimes_K B)^{\mathscr{D}} \simeq (A \otimes_K K^{\mathscr{D}}) \otimes_{K^{\mathscr{D}}} B^{\mathscr{D}}$$

as $K^{\mathscr{D}}$-algebras.

Since $\mathcal{F}$ is finitely-generated, we may choose a finitely-generated $K^{\mathscr{D}}$-subalgebra B' of $B^{\mathscr{D}}$ such that $\mathcal{F}$ is a subalgebra of $(A \otimes_K K^{\mathscr{D}}) \otimes_{K^{\mathscr{D}}} B'$. Since B is PI, $B^{\mathscr{D}}$ is PI, and B' is PI. Shirshov's height theorem implies that B' has polynomial growth (or, in other words, its Gelfand-Kirillov dimension is finite; see, for example, proof of Theorem 9.19 in [KR]). As $A \otimes_K K^{\mathscr{D}}$ is finite-dimensional (over $K^{\mathscr{D}}$), the tensor product $(A \otimes_K K^{\mathscr{D}}) \otimes_{K^{\mathscr{D}}} B'$ has polynomial growth too. But this contradicts the fact that its subalgebra $\mathcal{F}$ has exponential growth. □

Needless to say, this proof, unlike those in [PS], as well as all the proofs of the full-fledged Regev's $A \otimes B$, is absolutely non-constructive, as it uses existence of an ultrafilter, and, therefore, axiom of choice.

Along the same lines one may treat, at least in some particular cases, a number of other well-known results from PI theory: commutativity of an ordered PI algebra; PIness of a finitely-graded algebra with PI "null component"; of an algebra with a group action whose fixed point subalgebra is PI; of a localization of a PI-algebra; of algebras with involution, etc.

One may try to apply the same reasoning to the known open problem: suppose an associative algebra R is represented as the vector space sum of its subalgebras: $R = A + B$. If A, B are PI, is it true that R is PI? (see [FGL] and [KP] with a transitive closure of references therein). It is easy to see that the operation of taking ultraproduct commutes with the operation of taking the vector space sum: $R^{\mathscr{D}} = A^{\mathscr{D}} + B^{\mathscr{D}}$. Consequently, the question can be reduced to the following one: is it possible that the vector space sum $A + B$ of two PI algebras can contain a free associative subalgebra? If $A + B$ is finitely-generated, the impossibility of this follows from the same growth argument as in the proof of Theorem 3.1. The difficulty, however, lies in the fact that it is not clear how to reduce the situation to a finitely-generated one, as the multiplication between A and B could be intertwined in a complicated way.

4. Application: algebras with same identities

In [R, §5], Razmyslov obtained results claiming that some classes of finite-dimensional algebras (e.g., prime over algebraically closed field) are uniquely determined by their identities. Another result in this direction:

Theorem 4.1. *Let $\mathscr{P}$ be a class of finite-dimensional algebras satisfying the following conditions:*

(i) *If $A, B \in \mathscr{P}$, A and B are defined over the same field, A is a subalgebra of B, and $Var(A) = Var(B)$, then $A = B$.*
(ii) *$\mathscr{P}$ is closed under elementary equivalence in the first-order two-sorted theory of pairs (algebra over a field, field).*
(iii) *$\mathscr{P}$ contains all finite-dimensional prime algebras.*

Then $\mathscr{P}$ satisfies the following strengthening of the condition (i): if $A, B \in \mathscr{P}$, A and B are defined over the same field, and $Var(A) = Var(B)$, then $A \simeq B$.

We stress that the ground field over which algebras in the class $\mathscr{P}$ are defined, is not fixed.

Proof. Let $A, B \in \mathscr{P}$, both defined over a field K, are such that $Var(A) = Var(B)$. Lemma 1.5, Corollary 1.4 and Corollary 3.3 imply that a relatively free algebra $\mathcal{F}$ in the variety $Var(A) = Var(B)$ embeds, as a K-algebra, into an algebra $A \otimes K^{\mathscr{D}}$ for some ultrafilter $\mathscr{D}$. Hence $K^{\mathscr{D}}\mathcal{F}$ is isomorphic, as a $K^{\mathscr{D}}$-algebra, to a subalgebra of $A \otimes K^{\mathscr{D}}$. It is easy to see that primeness of the K-algebra $\mathcal{F}$ implies primeness of the $K^{\mathscr{D}}$-algebra $K^{\mathscr{D}}\mathcal{F}$. By (iii), the latter algebra belong to $\mathscr{P}$, and by Łoś' theorem and (ii), the $K^{\mathscr{D}}$-algebra $A \otimes_K K^{\mathscr{D}}$ belongs to $\mathscr{P}$. Obviously, $K^{\mathscr{D}}\mathcal{F}$ satisfies over $K^{\mathscr{D}}$ the same identities as A over K. Then by (i), $K^{\mathscr{D}}\mathcal{F} \simeq A \otimes_K K^{\mathscr{D}}$. By the same reasoning, $K^{\mathscr{D}}\mathcal{F} \simeq B \otimes_K K^{\mathscr{D}}$. Hence $A \otimes_K K^{\mathscr{D}} \simeq B \otimes_K K^{\mathscr{D}}$ as $K^{\mathscr{D}}$-algebras, and $A \simeq B$ as K-algebras. □

This theorem allows, for example, to obtain an alternative proof of Razmyslov's results in an important particular case of finite-dimensional simple Lie algebras over an algebraically closed field of characteristic zero (see [R, §5, Corollary 1 and Comments]). Indeed, the condition of the theorem are fulfilled for such class of algebras: (i) can be proved with the help of the well-known Dynkin's classification [D] of semisimple subalgebras of semisimple Lie algebras, (ii) is evident, and (iii) follows from the obvious fact that for finite-dimensional Lie algebras over a field of characteristic zero, simplicity is equivalent to primeness.

The same approach could be applied to finite-dimensional simple Jordan algebras (what follows from the general Razmyslov's results and also established independently in [DR]).

Along the same lines one may treat varieties generated by affine Kac–Moody algebras. Consider, for example, a Lie algebra of the form

$$\widehat{\mathfrak{g}} = \mathfrak{g} \otimes K[t, t^{-1}] + Kz, \tag{5}$$

where $\mathfrak{g}$ is a split finite-dimensional simple Lie algebra defined over a field K of characteristic zero, z is the central element, and the multiplication between elements of $\mathfrak{g} \otimes K[t, t^{-1}]$ is twisted by the well-known 2-cocycle:

$$[x \otimes f, y \otimes g] = [x, y] \otimes fg + (x, y)Res(\frac{df}{dt}g)z$$

where $x, y \in \mathfrak{g}$, $f, g \in K[t, t^{-1}]$, and $(\cdot , \cdot)$ is the Killing form on $\mathfrak{g}$.

In [Z] it is shown, among other, that $Var(\widehat{\mathfrak{g}}) = [Var(\mathfrak{g}), \mathbf{E}]$, where $\mathbf{E}$ is the variety consisting of the single zero algebra, and $[\cdot , \cdot]$ is the standard commutator of varieties as defined in [Ba, §4.3.8] (in other words, for a variety $\mathbf{V}$, $[\mathbf{V}, \mathbf{E}]$ is nothing but a variety defined by identities of the form $[f(x_1, \dots, x_n), x_{n+1}] = 0$, where $f(x_1, \dots, x_n) = 0$ is an identity in $\mathbf{V}$). Let us complement this result by showing that relatively free algebras in $Var(\widehat{\mathfrak{g}})$ embed in algebras whose structure closely resembles those of $\widehat{\mathfrak{g}}$.

Note that the Lie algebra $\mathfrak{g} \otimes K[t, t^{-1}]$ is simple, so $\widehat{\mathfrak{g}}$ is a central extension of a simple (and hence, prime) Lie algebra. By an obvious modification of the proof of Lemma 1.5, one get that for the countably-generated relatively free algebra $\mathcal{L}$ in $Var(\widehat{\mathfrak{g}})$, $\mathcal{L}/Z(\mathcal{L})$ is prime, and, by Theorem 1.2 and Lemma 3.2, embeds in $\mathfrak{g} \otimes K[t, t^{-1}]^{\mathscr{D}}$ for a certain ultrafilter $\mathscr{D}$. From the results of [Ba, §4.4] it follows that $\mathcal{L}$ embeds in a central extension of $\mathfrak{g} \otimes K[t, t^{-1}]^{\mathscr{D}}$. The latter, by [K, Theorem 3.3 and Corollary 3.5], is described in terms of the first-order cyclic homology of $K[t, t^{-1}]^{\mathscr{D}}$, so we get an imbedding

$$\mathcal{L} \subset \mathfrak{g} \otimes K[t, t^{-1}]^{\mathscr{D}} + HC_1(K[t, t^{-1}]^{\mathscr{D}}),$$

where the multiplication in the right-hand side Lie algebra is defined by the formula

$$[x \otimes F, y \otimes G] = [x, y] \otimes FG + (x, y)\overline{F \wedge G},$$

where $x, y \in \mathfrak{g}$, $F, G \in K[t, t^{-1}]^{\mathscr{D}}$, and $\overline{F \wedge G}$ denotes the corresponding homology class in $HC_1(K[t, t^{-1}]^{\mathscr{D}})$.

The addition in (5) of the $Kt\frac{d}{dt}$ term, or twisting by automorphisms of $\mathfrak{g}$, do not significantly change the picture, and could be treated in the same way.

5. Application: Tarski's monsters

Under *Tarski's monster of type p*, p being a prime (respectively, of type ∞) we understand an infinite nonabelian group all whose proper subgroups are cyclic of order p (respectively, of infinite order). Such groups were constructed, among other groups with exotic-looking restrictions on subgroups, by Olshanskii in the framework of his celebrated machinery of geometrically-motivated manipulations with group presentations (see, for example, [O1, Chapter 9, §28.1]).

Let G be a finitely-generated group, and

$$\{1\} \to \mathcal{N} \to \mathcal{F} \to G \to \{1\} \tag{6}$$

is its presentation, where $\mathcal{F}$ is a finitely-generated free group, and $\mathcal{N}$ is its normal subgroup of relations. The *girth* of the presentation (6) is the minimal length of elements of $\mathcal{N}$, i.e., the minimal length of relations between the chosen generators of G (or, in other words, the minimal length of a simple loop in the corresponding Cayley graph). The *girth* of G is the supremum of girths of all its presentations with a finite number of generators. This natural notion was introduced and studied recently by Akhmedov in [A1] and [A2], and by Schleimer in [S]. One of the interesting questions arising in that regard is to construct groups of infinite girth.

As noted in the above-mentioned works, a group satisfying a nontrivial identity cannot have an infinite girth. To circumvent this obstacle, let us introduce the notion of *relative girth* – a girth relative to all identities a group satisfies: in the definition of girth above, replace in (6) the (absolutely) free group $\mathcal{F}$ by a (finitely-generated) relatively free group in the variety $Var(G)$.

Theorem 5.1.

(i) *A Tarski's monster of type p does not satisfy any nontrivial identity except* $x^p = 1$ *and its consequences, if and only if it has infinite relative girth.*

(ii) *A Tarski's monster of type* ∞ *does not satisfy any nontrivial identity if and only if it has infinite girth.*

Remark (A. Olshanskii). Tarski's monsters satisfying the condition (ii) of the theorem do exist (and, moreover, there is an abundance of them), and they can be constructed in the following way.

According to [O2, Corollary 1], each non-cyclic torsion-free hyperbolic group G_0 has a homomorphic image G which is a Tarski's monster of type ∞. Such monsters are constructed by a subsequent application of [O2, Theorem 2], as the direct limit of a system of surjective maps of groups $G_0 \to G_1 \to \dots$. Each G_n is a non-cyclic torsion-free hyperbolic group (and hence contains a countably-generated free subgroup), and is 2-generated for $n \geq 1$. Let us denote, by abuse of notation, the corresponding generators by the same letters a, b (so, $a, b \in G_n$ are images of $a, b \in G_{n-1}$), and each G_n is obtained from G_{n-1} by adding additional relations between these two generators. Also, the injectivity radius of each surjection $G_{n-1} \to G_n$ (i.e., the maximal number r such that the map is injective on all words of length $\leq r$) can be chosen to be arbitrarily large.

Enumerate all non-trivial words in the free countably-generated group $\mathcal{F}$ as $v_1, v_2, \dots$. Since each G_n contains a copy of $\mathcal{F}$, there are elements in G_n such that the value of v_n on these elements is different from 1. Writing these elements in terms of the generators a, b, we get

$$w_n(a,b) = v_n(w_{n1}(a,b), w_{n2}(a,b), \dots) \neq 1 \text{ in } G_n \tag{7}$$

for some (finite number of) words $w_n, w_{n1}, w_{n2}, \dots$.

Now, on each step choose the injectivity radius of the surjection $G_n \to G_{n+1}$ larger than the length of all words $w_1, \dots, w_n$ constructed in the previous steps. Consequently, (7) holds in all groups $G_{n+1}, G_{n+2}, \dots$, and hence in the limit group G. This implies that $v_n = 1$, for any n, cannot be an identity of G.

Question 5.1. *Prove existence of Tarski's monsters satisfying the condition (i) of Theorem 5.1.*

Proof of Theorem 5.1. The "only if" part is obvious, so let us prove the "if" part. Let G be Tarski's monster either of type p which does not satisfy any nontrivial identity except $x^p = 1$ and its consequences, or of type ∞ which does not satisfy a nontrivial identity. By Corollary 2.2, a group elementary equivalent to G contains a subgroup isomorphic to a relatively free 2-generated subgroup $\mathcal{G}$ (which is the free Burnside group $B(2,p)$ in the case (i), or the free group in the case (ii)). Let x, y be the free generators of $\mathcal{G}$, and

$$\{w_1(x,y) = x, w_2(x,y) = x^{-1}, w_3(x,y) = y, w_4(x,y) = y^{-1}, \dots, w_{k_n}(x,y)\}$$

is the set of all words of $\mathcal{G}$ of length $\le n$. The existence of this "initial piece of $\mathcal{G}$ of length n" can be written as the first-order property:

$$\exists x \; \exists y : \bigwedge_{1 \le i < j \le k_n} w_i(x,y) \ne w_j(x,y).$$

Consequently, for each $n \in \mathbb{N}$, the same first-order formula holds in G, let $x_n, y_n \in G$ be the corresponding elements. Obviously, x_n, y_n do not commute except, possibly, for some small values of n, and, therefore, generate G. This provides a presentation of G of (relative) girth $> n$. □

Note another interesting consequence of Theorem 5.1.

The *growth sequence* of a group G is a sequence whose nth term equal to the (minimal) number of generators of the nth fold direct power of G. See [W] for a brief history of the subject and further references. In particular, in a number of works, including [W], a considerable effort was put into construction of groups whose growth sequence is constant, each term is equal to 2. Theorem 5.1 provides further such examples, in view of the following general elementary fact:

Lemma 5.2. *If a finitely-generated simple group G has infinite relative girth, then its growth sequence is constant, each term is equal to the minimal number of generators of G.*

Proof. Let n be the minimal number of generators of G. Infinity of the relative girth of G means that there is an infinite sequence $\mathcal{N}_1, \mathcal{N}_2, \dots$ of normal subgroups of the free n-generated relatively free group $\mathcal{G}$ in $Var(G)$ such that for every i the length of each word in $\mathcal{N}_i$ is $\ge i$, and $\mathcal{G}/\mathcal{N}_i \simeq G$.

Let us prove by induction that for each $k \in \mathbb{N}$ there is a sequence $i_1 < i_2 < \dots < i_k$ such that

$$\mathcal{G}/(\mathcal{N}_{i_1} \cap \dots \cap \mathcal{N}_{i_k}) \simeq G \times G \times \dots \times G \quad (k \text{ times}). \tag{8}$$

For $k = 1$ we may take $i_1 = 1$. Suppose that for some $k > 1$ the isomorphism (8) holds. It is obvious that $\mathcal{N}_{i_1} \cap \dots \cap \mathcal{N}_{i_k} \ne \{1\}$. On the other hand, since $\bigcap_{i > i_k} \mathcal{N}_i = \{1\}$, there is $i_{k+1} > i_k$ such that

$$\mathcal{N}_{i_1} \cap \dots \cap \mathcal{N}_{i_k} \not\subseteq \mathcal{N}_{i_{k+1}}.$$

Since G is simple, $\mathcal{N}_{i_{k+1}}$ is a maximal normal subgroup in $\mathcal{G}$, and

$$\mathcal{G} = (\mathcal{N}_{i_1} \cap \dots \cap \mathcal{N}_{i_k})\mathcal{N}_{i_{k+1}}.$$

Then:

$$\begin{aligned}\mathcal{G}/(\mathcal{N}_{i_1} \cap \dots \cap \mathcal{N}_{i_k} \cap \mathcal{N}_{i_{k+1}}) &\simeq \mathcal{G}/(\mathcal{N}_{i_1} \cap \dots \cap \mathcal{N}_{i_k}) \times \mathcal{G}/\mathcal{N}_{k+1} \\ &\simeq G \times \dots \times G \quad (k+1 \text{ times}).\end{aligned}$$

It follows from (8) that each finite direct power of G is n-generated. □

In fact, nothing in this proof is specific to groups: the corresponding statement could be formulated for general algebraic systems; in particular, it holds also for algebras.

Lemma 5.2 implies that the growth sequence of Tarski's monsters satisfying the conditions of Theorem 5.1 is constant, each term is equal to 2.

6. Further speculations

Here we indicate some of our initial reasons for looking into all this, of a highly speculative character.

6.1. **Lie-algebraic monsters.** A problem of existence of Lie-algebraic analogs of Tarski's monsters – namely, of infinite-dimensional Lie algebras all whose proper subalgebras are one-dimensional – is, arguably, one of the most difficult problems in the abstract theory of infinite-dimensional Lie algebras. For such hypothetical Lie algebras which do not satisfy a nontrivial identity, an analog of Theorem 5.1 would hold.

Question 6.1. *Study the notion of (relative) girth for Lie algebras.*

6.2. **Tits' alternative.** Another reason was the desire to provide an alternative proof of the celebrated Tits' alternative, or for one of its not less celebrated consequences, such as growth dichotomy for linear group. Tits' alternative claims that a linear group contains either a solvable subgroup of finite index, or a nonabelian free subgroup. A nice and important, yet admitting a few-lines elementary proof, result of Platonov [P] states that a linear group which has a nontrivial identity, contains a solvable subgroup of finite index. Modulo this result, the proof of Tits' alternative reduces to establishing that a linear group G which does not satisfy a nontrivial identity, contains a nonabelian free subgroup. One may naively argue as follows: by Corollary 2.3, a group, elementary equivalent to G, contains a nonabelian free subgroup. From this we may infer some first-order properties of G, for example, that it contains "a piece of a nonabelian free group of arbitrarily large length", like in the proof of Theorem 5.1. On the other hand, as linearity is a sort of finiteness condition, one may hope that these first-order properties may help to construct a nonabelian free subgroup in G. If successful, this approach would provide a proof of the Tits' alternative drastically different from all the proofs given so far.

6.3. **Jacobson's problem.** An old open problem due to Jacobson asks whether a Lie p-algebra L such that for every $x \in L$ there is $n(x) \in \mathbb{N}$ satisfying

$$x^{p^{n(x)}} = x, \tag{9}$$

is abelian? As one of the first steps, one may wish to prove that such Lie algebras satisfy a nontrivial identity; or, for example, that there are no simple (or even prime) such algebras. In both of these cases, by Lemma 1.5 and Corollary 1.4, a free Lie algebra embeds in an algebra elementary equivalent to L, over some elementary extension of the ground field. The condition (9) is not the first-order property (unless all $n(x)$ are bounded, in which case the problem is trivial), but one may hope to derive from it some first-order consequences which will come into contradiction with the existence of a free Lie subalgebra.

Acknowledgements

Thanks are due to Misha Gavrilovich, Sergei Ivanov, Mark Sapir, and especially to Alexander Olshanskii, for very helpful comments.

References

[A1] A. Akhmedov, *On the girth of finitely generated groups*, J. Algebra **268** (2003), 198–208.

[A2] ———, *The girth of groups satisfying Tits alternative*, J. Algebra **287** (2005), 275–282.

[Ba] Yu.A. Bahturin, *Identical Relations in Lie Algebras*, VNU Sci. Press, 1987.

[BS] J.L. Bell and A.B. Slomson, *Models and Ultraproducts*, 2nd revised printing, North-Holland, 1971.

[Bo] M. Boffa, *Elimination of inverses in groups*, Model Theory of Groups and Automorphism Groups (ed. D.M. Evans), Lond. Math. Soc. Lect. Note Ser. **244** (1997), 134–143.

[DR] V.S. Drensky and M.L. Racine, *Distinguishing simple Jordan algebras by means of polynomial identities*, Comm. Algebra **20** (1992), 309–327.

[DS] C. Druţu and M. Sapir (with an appendix by D. Osin and M. Sapir), *Tree-graded spaces and asymptotic cones of groups*, Topology **44** (2005), 959–1058; `arXiv:math/0405030`.

[D] E.B. Dynkin, *Semisimple subalgebras of semisimple Lie algebras*, Mat. Sbornik **30** (1952), 349–462 (in Russian); translated in AMS Transl. **6** (1957), 245–378; reprinted in *Selected Papers of E.B. Dynkin with Commentary*, AMS, 2000, 175–244.

[E] P. Eklof, *Ultraproducts for algebraists*, Handbook of Mathematical Logic (ed. J. Barwise), North-Holland, 1977, 105–137.

[FGL] B. Felzenszwalb, A. Giambruno and G. Leal, *On rings which are sums of two PI-subrings: a combinatorial approach*, Pacific J. Math. **209** (2003), 17–30.

[KR] A. Kanel-Belov and L.H. Rowen, *Computational Aspects of Polynomial Identities*, AK Peters, 2005.

[K] C. Kassel, *Kähler differentials and coverings of complex semisimple Lie algebras extended over a commutative algebra*, J. Pure Appl. Algebra **34** (1984), 265–275.

[KP] M. Kępczyk and E.R. Puczyłowski, *On the structure of rings which are sums of two subrings*, Arch. Math. **83** (2004), 429–436.

[MM] L. Makar-Limanov and P. Malcolmson, *Free subalgebras of enveloping fields*, Proc. Amer. Math. Soc. **111** (1991), 315–322.

[NN] A. Niţă and C. Niţă, *Sur l'ultraproduit des anneaux de matrices*, Bull. Math. Soc. Sci. Math. Rep. Soc. Roumanie **29** (1985), 275–278.

[O1] A.Yu. Olshanskii, *The Geometry of Defining Relations in Groups*, Kluwer, 1991.

[O2] ———, *On residualing homomorphisms and G-subgroups of hyperbolic groups*, Intern. J. Algebra Comput. **3** (1993), 365–409.

[P] V.P. Platonov, *Linear groups with identity relations*, Doklady AN BSSR **11** (1967), 581–582 (in Russian).

[PS] C. Procesi and L. Small, *Endomorphism rings of modules over PI-algebras*, Math. Z. **106** (1968), 178–180.

[R] Yu.P. Razmyslov, *Identities of Algebras and Their Representations*, AMS, 1994.

[S] S. Schleimer, *On the girth of groups*, Preprint, 2000; `http://math.rutgers.edu/~saulsch/Maths/girth.pdf` .

[T] F. Taha, *Algèbres simples centrales sur les corps ultra-produits de corps p-adiques*, Séminaire d'Algèbre Paul Dubreil et Marie-Paul Malliavin (Paris 1981), Lect. Notes Math. **924** (1982), 89–128.

[W] D.T. Wise, *The rank of a direct power of a small-cancellation group*, Geom. Dedicata **94** (2002), 215–223.

[Z] M.V. Zaicev, *Varieties and identities of affine Kac-Moody algebras*, Methods in Ring Theory (ed. V. Drensky, A. Giambruno and S. Sehgal), Lecture Notes Pure Appl. Math. **198** (1998), 303–314.

E-mail address: justpasha@gmail.com