Arnol’d, the Jacobi Identity, and Orthocenters

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Abstract. The three altitudes of a plane triangle pass through a single point, called the orthocenter of the triangle. This property holds literally in Euclidean geometry, and, properly interpreted, also in hyperbolic and spherical geometries. Recently, V. I. Arnol’d offered a fresh look at this circle of ideas and connected it with the well-known Jacobi identity. The main goal of this article is to present an elementary version of Arnol’d’s approach. In addition, several related ideas, including ones of M. Chasles, W. Fenchel and T. Jørgensen, and A. A. Kirillov, are discussed.

To the memory of Vladimir Igorevich Arnol’d

1. INTRODUCTION. The three altitudes of a Euclidean triangle are concurrent, i.e., pass through a single point, called the orthocenter of the triangle. This theorem was known to Euclid and did not escape the attention of Euler and Gauss. For Euler’s proof of this theorem, see [7, Chapter 1, Section 7], and for Gauss’s proof see [4, Exercise II.18], for example.

The corresponding theorem is not true in hyperbolic geometry, if interpreted in the most direct way. Sometimes no two altitudes of a hyperbolic triangle intersect. But if two altitudes do intersect at a point, then the third one also passes through this point, which is then called the orthocenter of the triangle. Moreover, if two altitudes are asymptotically parallel (i.e., have a common point at infinity), the third one is asymptotically parallel to both of them, and if two altitudes have a common perpendicular, the third altitude is also orthogonal to it. (Recall that any two lines in a hyperbolic plane either intersect, or are asymptotically parallel, or have a unique common perpendicular.) One may consider these three statements together as the theorem about the altitudes in hyperbolic geometry. A synthetic proof of this result is outlined in [10, Exercise 40.14].

In fact, one can make sense of the orthocenter even in the case when the altitudes do not intersect. In this case the orthocenter lies outside the hyperbolic plane, sometimes on the circle at infinity (when the altitudes are asymptotically parallel), but usually beyond it (see Remark 1 at the end of Section 6).

Recently, V. I. Arnol’d [3] offered a fresh look at this circle of issues. Namely, he showed that the Jacobi identity

\[[A, B], C] + [[B, C], A] + [[C, A], B] = 0\]

lies at the heart of the theory of altitudes in hyperbolic geometry. In his approach, Arnol’d used the Jacobi identity for the Poisson bracket of quadratic forms on \(\mathbb{R}^2\) endowed with its canonical symplectic structure. Unfortunately, the use of these advanced notions renders Arnol’d’s approach nonelementary.

The main goal of this article is to present an elementary version of Arnol’d’s ideas. We use the most simple form of the Jacobi identity, namely, the Jacobi identity for the commutator \([A, B] = AB - BA\) of 2-by-2 matrices. This allows us to improve upon

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Arnol’d’s exposition in another respect: while Arnol’d needs to bring his quadratic forms to diagonal form in order to do his computations, we do not need to bring our matrices (which play a role similar to the role of quadratic forms in Arnol’d’s approach) to any normal form. In addition, we prove a sufficient condition for the altitudes to intersect, and show that the altitudes sometimes do not intersect if this condition is violated (see Section 7). The latter results are merely stated by Arnol’d (probably, because they are comparatively simple). In Section 8 we prove the theorem about the altitudes in spherical geometry by using the Jacobi identity for the usual vector product in 3-space. Sections 9 and 10 are devoted to the Euclidean case. While the theorem is simpler in the Euclidean case, its connection with the Jacobi identity is not so direct as in the hyperbolic and spherical cases (see the discussion at the end of Section 10). Sections 11 and 12 are devoted to an alternative approach to the hyperbolic altitudes theorem, still based on the Jacobi identity. This approach, based on Fenchel’s theory of lines [9], is less elementary, but is very elegant nonetheless. In Section 12 we compare this approach with the approach of Sections 2–6.

The reader familiar with Arnol’d’s paper may notice that our approach bypasses a point made by Arnol’d, namely, that these geometric theorems are intimately connected to mathematical physics. Instead, in this note these theorems are related to a little piece of algebra (the algebra of 2-by-2 matrices). Here I would like to quote A. Weil (see [19, Preface]).

We are dealing here with mathematics, not with theology.... I have tried to show that, from the point of view which I have adopted, one could give a coherent treatment, logically and aesthetically satisfying, of the topics I was dealing with.

The prerequisites for reading this article are rather modest. We expect that the reader has had no more than a fleeting encounter with hyperbolic geometry and the projective plane. Since the Poincaré unit disc model is the most ubiquitous version of hyperbolic geometry, we choose it as our starting point. We will review the basic facts about these topics along the way. On the algebraic side we expect that the reader is familiar with symmetric bilinear forms (pairings) and matrices. Only in the last two Sections, 11 and 12, will we assume more from the reader (namely, familiarity with the upper half-space model of hyperbolic 3-space).

Most of our arguments are carried out in the less well-known projective Klein model, and we explain its definition, basic properties, and relation to the Poincaré model in Section 2. In order to relate the Poincaré and Klein models we will use the basic properties of inversions, which can be found in [7, Chapter 6]. A crucial tool in our arguments will be the notion of polarity, discussed from the geometric and algebraic points of view in Sections 3 and 4 respectively.

We use polarity in a manner similar to that of Arnol’d. In fact, the use of polarity in order to prove the hyperbolic altitudes theorem goes back more than one hundred years. J. L. Coolidge (see [5, Theorem 2, p. 103]) included such a proof in his classical treatise [5]. Another classical proof based on polarity is presented by Coxeter [6, Section 11.6]. The approach of Coolidge is closer in spirit to that of Arnol’d and of the present paper, but neither he nor Coxeter relate the theorem to the Jacobi identity.

The heart of the paper is Section 6, where we use the Jacobi identity to prove the altitudes theorem in hyperbolic geometry, after an important preliminary observation is made in Section 5.

Sections 7–12 complement this result in several ways, as described above. These sections are independent from each other, with the exception of Section 12, which depends on Section 11.
2. PRELIMINARIES. The Poincaré unit disc model of (plane) hyperbolic geometry is the open unit disc \( U = \{(x, y) : x^2 + y^2 < 1\} \subset \mathbb{R}^2 \). The points of the model are the points of this unit disc, and its lines are the intersections of the unit disc with the circles orthogonal (in the usual Euclidean sense) to its boundary (the unit circle), and the diameters of the disc without their endpoints. These diameters are usually regarded as the intersections of the unit disc with the circles of infinite radius (i.e., the Euclidean lines) orthogonal to the unit circle.

The angles between these hyperbolic lines are by definition equal to the angles between them in the Euclidean sense. One can define hyperbolic distances and areas, but we will be concerned only with angles, and, except in Section 7, with right angles. For this reason we do not discuss the metric aspects of the Poincaré model.

Now let us introduce the projective Klein model of hyperbolic geometry. We will use the upper hemisphere \( S^2_+ = \{(x, y, z) : x^2 + y^2 + z^2 < 1, z > 0\} \subset \mathbb{R}^3 \) as an intermediary to pass from the Poincaré model to the Klein one. Let \( V \) be the orthogonal projection of the upper hemisphere \( S^2_+ \) to the equatorial disc \( \{(x, y) : x^2 + y^2 < 1, z = 0\} \), which we will identify with \( U \). In other words, \( V(x, y) = (x, y) \). Let \( S \) be the stereographic projection of the upper hemisphere \( S^2_+ \) to the equatorial disc from the south pole \( s = (0, 0, -1) \). The map \( S \) is the restriction of the stereographic projection \( \hat{S} : S^2 \setminus \{s\} \to \mathbb{R}^2 \). By definition, \( \hat{S}(p) \) is equal to the point of intersection of the line connecting \( s \) with \( p \) and the plane \( \{(x, y, z) \in \mathbb{R}^3 : z = 0\} \), which we identify with \( \mathbb{R}^2 \).

**Proposition.** \( \hat{S} \) preserves angles and takes circles on \( S^2 \) (which are the intersections of \( S^2 \) with Euclidean planes) to circles in the plane \( \mathbb{R}^2 \).

**Proof.** A convenient way to prove this is to use the inversion \( I \) of \( \mathbb{R}^3 \) with respect to the sphere with center \( s \) and radius \( \sqrt{2} \). Recall that for any point \( p \in \mathbb{R}^3 \setminus \{s\} \), the image \( I(p) \) is defined to be the point \( q \) on the ray from \( s \) through \( p \) such that the product of the distances from \( p \) and \( q \) to \( s \) is equal to \( (\sqrt{2})^2 \).

Let us prove first that \( I \) takes \( S^2 \) into \( \mathbb{R}^2 \). Since \( S^2 \) contains \( s \), the inversion \( I \) takes \( S^2 \setminus \{s\} \) into a plane \( P \). Note that the distance from \( s \) to the points on the equator of \( S^2 \) (i.e., the intersection of \( S^2 \) with the plane \( z = 0 \)) is equal to \( \sqrt{2} \), and therefore these points are fixed by \( I \). So \( P \) must contain the equator of \( S^2 \), and therefore it must be the plane \( z = 0 \).

Now it is easy to see that \( I \) restricted to \( S^2 \setminus \{s\} \) is equal to the stereographic projection \( \hat{S} \). Let \( l \) be any line intersecting \( S^2 \) at \( s \) and some other point \( p \). Let \( q \) be the point of intersection of \( l \) with \( P = \mathbb{R}^2 \). Since \( I \) preserves \( l \) (because \( l \) contains \( s \)) and takes \( S^2 \) to \( \mathbb{R}^2 \), we see that \( I(p) = q \). Therefore, \( I \) restricted to \( S^2 \) is equal to the stereographic projection \( \hat{S} \).

Now we can deduce the properties of \( \hat{S} \) from the standard properties of \( I \). Namely, it is well known that inversions take circles to circles (lines are considered as circles passing through infinity) and preserve angles. (In three dimensions one may establish the first fact by using the facts that circles are intersections of spheres and planes, and inversions take spheres and planes into spheres and planes.)

**Corollary.** The map \( S^{-1} \) takes lines in the Poincaré model into vertical (i.e., orthogonal to \( \mathbb{R}^2 \)) semicircles.

**Proof.** Note that \( \hat{S} \) fixes (pointwise) the equator \( S^1 = \{(x, y, 0) : x^2 + y^2 = 1\} \) of \( S^2 \). By the proposition, \( \hat{S} \) takes arcs of circles orthogonal to \( S^1 \) and having endpoints on \( S^1 \) into arcs of circles on \( S^2 \) orthogonal to \( S^1 \) (and having endpoints on \( S^1 \)). The corollary follows.
Corollary. The map $V \circ S^{-1}$ takes lines in the Poincaré model to chords of the unit circle.

Proof. The vertical projections of vertical semicircles are chords. 

Now we can define the Klein model of hyperbolic geometry. It is the image of the Poincaré model under the map $V \circ S^{-1}$. So, the points of the Klein model are the points of the unit disc, as before, but the lines are the chords of the unit circle. The angle between two lines is defined as the angle between the corresponding lines (arcs of circles) in the Poincaré model. In contrast with the Poincaré model, this angle usually does not agree with the Euclidean angle between these lines. In the next section we will introduce the notion of polarity, which will allow us to recognize the orthogonality of lines in the Klein model, i.e., right angles.

Note that (since both $V$ and $\hat{S}^{-1}$ fix $S^1$), the line in the Klein model corresponding to a line in the Poincaré model is the chord with the same endpoints as the Poincaré line (an arc of a circle).

3. POLARITY FROM THE GEOMETRIC POINT OF VIEW. For every chord $C$ of the unit circle $S^1$ we will define a point $p$ outside of $S^1$, which we will call the point (geometrically) polar to $C$. If $C$ is not a diameter of $S^1$, we define $p$ as the point of intersection of the two lines tangent to $S^1$ at the endpoints of $C$.

In order to make this definition work for diameters also, we embed the Euclidean plane $\mathbb{R}^2$ in the projective plane $\mathbb{P}^2$ by adding points at infinity in the usual way. Namely, for every line $k$ in $\mathbb{R}^2$ let us denote by $[k]$ the set of all lines parallel to $k$. For every such set $[k]$ we add to $\mathbb{R}^2$ a new point, which we may denote also by $[k]$. The result is the projective plane $\mathbb{P}^2$. The collection of added points $[k]$ forms a new line, called the line at infinity. We extend every line $k$ by adding to it the point $[k]$ at infinity. Then any two parallel lines intersect at a well-defined point at infinity (and the line at infinity intersects any other line at a single point).

Since the lines tangent to $S^1$ at the endpoints of a diameter are parallel, they intersect at one point at infinity. In other words, the polar point $p$ of a diameter $d$ is a well-defined point of $\mathbb{P}^2$ and lies on the line at infinity. It is equal to $[k]$, where $k$ is any line in $\mathbb{R}^2$ orthogonal (in the Euclidean sense) to $d$. Clearly, any point outside of the unit circle $S^1$ (including points at infinity) is polar to a unique chord of $S^1$.

Now we can give a criterion for two lines in the Klein model to be orthogonal (in the sense of the Klein model). For a line $l$ in the Klein model we will denote by $\overline{l}$ the projective line extending $l$.

Theorem 1. Two lines $l$ and $m$ in the Klein model are orthogonal if and only if $\overline{m}$ contains the point polar to $l$ (or, equivalently, $\overline{l}$ contains the point polar to $m$).

Proof. See Figure 1. Let $p$ and $q$ be the points polar to $l$ and $m$, respectively. Let $L$ and $M$ be the Poincaré lines corresponding to $l$ and $m$, respectively. So, $L$ is the arc of a circle $\overline{L}$ having the same endpoints as $l$ and orthogonal to $S^1$. It follows that the center of $\overline{L}$ is the polar point $p$. Similarly, $M$ is an arc of a circle $\overline{M}$ with center $q$.

By definition, $l$ is orthogonal to $m$ in the sense of the Klein model if and only if $L$ is orthogonal to $M$ in the sense of the Poincaré model, i.e., in the usual Euclidean sense.

Let us consider the inversion $I$ in the circle $\overline{L}$. It preserves all lines passing through $p$, in particular, the two such lines tangent to $S^1$. Since any inversion takes circles to circles, $I$ takes $S^1$ into another circle tangent to these two lines. Since $I$ fixes the circle $\overline{L}$ (pointwise), and, in particular, its intersection with $S^1$ (which consists of the two
endpoints of $L$), it follows that $I$ takes $S^1$ to $S^1$ (but does not fix it pointwise). Also, $I$ fixes the two points of intersection of $L$ and $M$.

Now, suppose that $l$ is orthogonal to $m$ in the sense of the Klein model. This means that $L$ is orthogonal to $M$. Then $M$ is orthogonal to $L$ (at one point of intersection, and, therefore, at the other), and $I(M)$ is a circle orthogonal to $L$ at these two points of intersection. It follows that $I$ preserves $M$.

Since the inversion $I$ also preserves $S^1$, it interchanges the two points of intersection of $M$ and $S^1$, i.e., the endpoints of $m$. This may happen only if $m$ contains $p$, as claimed.

Conversely, suppose that $m$ contains $p$. Then $I$ interchanges the two intersection points of $M$ and $S^1$. Since the circle $M$ is orthogonal to $S^1$, it follows that its image is equal to $M$ (there is only one circle orthogonal to $S^1$ at two given points). It follows that $M$ is orthogonal to $L$ (because at a point of intersection of $M$ and $L$ the inversion $I$ reflects the tangent line to $M$ with respect to the tangent line of $L$).

If $l$ is a diameter of $S^1$, the point $p$ lies at infinity. In this case we should take as $I$ the reflection in the line $L$, and then the rest of the proof is similar. Also, if $m$ is a diameter, then the circle $M$ is actually a line (equal to $m$). The reader can either consider these special cases separately, or just treat the lines as circles with center at infinity.

The above proof closely follows the proof of this theorem in [18, Section 4.8].

4. POLARITY FROM THE ALGEBRAIC POINT OF VIEW. Let us recall the standard description of the projective plane $P^2$ in terms of homogeneous coordinates. Call two nonzero triples of real numbers $(x, y, z)$ and $(x', y', z')$ equivalent if $(x', y', z') = (\lambda x, \lambda y, \lambda z)$ for some nonzero real number $\lambda$. We will denote by $[x : y : z]$ the equivalence class of the triple $(x, y, z)$. One may consider $P^2$ to be the set of such equivalence classes. Indeed, if $z \neq 0$, then the equivalence class $[x : y : z]$ contains a unique representative of the form $(x, y, 1)$, and we identify it with the point $(x, y) \in \mathbb{R}^2$. The points of the form $[x : y : 0]$ form the line at infinity. Every line in $\mathbb{R}^2$ is given by an equation of the form $ax + by + c = 0$, where at least one of $a$ and $b$ is nonzero. In order to form the corresponding line in $P^2$ we add to it the point $[b : -a : 0]$; the resulting line is given by the homogeneous equation
ax + by + cz = 0 (i.e., it consists of classes [x : y : z] such that (x, y, z) satisfies this equation). The line at infinity is given by any equation of the form cz = 0 with c \neq 0. So, all homogeneous equations of the form ax + by + cz = 0, where at least one of a, b, and c is nonzero, define lines in \( P^2 \), and every line has this form. The triple \((a, b, c)\) is determined by the line up to multiplication by a nonzero \( \lambda \in \mathbb{R} \).

Recall that a pairing on a real vector space \( V \) (over \( \mathbb{R} \)) is a bilinear map \( V \times V \to \mathbb{R} \). The key role in the algebraic approach to polarity is played by the pairing on \( \mathbb{R}^3 \) defined by the formula

\[
\langle (x, y, z), (x', y', z') \rangle = xx' + yy' - zz'.
\]

This pairing is obviously symmetric, i.e., \( \langle p, q \rangle = \langle q, p \rangle \) for all \( p, q \in \mathbb{R}^3 \). Let \( Q(a) = \langle a, a \rangle \), where \( a \in \mathbb{R}^3 \). We call \( Q \) the quadratic form associated to the pairing \( \langle \cdot, \cdot \rangle \).

In homogeneous coordinates the unit circle \( S^1 \) is given by the equation \( x^2 + y^2 - z^2 = 0 \). Indeed, this equation has no nonzero solutions with \( z = 0 \), so its set of solutions is actually contained in \( \mathbb{R}^2 \). Written as an equation for the points \([x : y : 1] \in \mathbb{R}^2\), it turns into the familiar equation \( x^2 + y^2 - 1 = 0 \) of the unit circle. In terms of the above quadratic form we can write this equation as \( Q(x, y, z) = 0 \).

In order to avoid cluttered notation, we will often abuse notation by not distinguishing between a point \( p \in \mathbb{P}^2 \) and its representatives in \( \mathbb{R}^3 \), i.e., between \([x : y : z]\) and \((x, y, z)\). Since we will do this only when our equations are homogeneous, this can cause no harm.

Let us define the line (algebraically) polar to a point \( a \in \mathbb{P}^2 \) as the projective line defined by the homogeneous equation \( \langle p, a \rangle = 0 \) for the point \( p \in \mathbb{P}^2 \). We will denote this line by \( a^\perp \). This is an example of abuse of notation alluded to in the previous paragraph; strictly speaking, we should write here representatives of \( a \) and \( p \) in \( \mathbb{R}^3 \) instead of \( a \) and \( p \). Note that if \( a^\perp = b^\perp \), then \( a = b \). Indeed, if equations \( \langle p, a \rangle = 0 \) and \( \langle p, b \rangle = 0 \) have the same solutions \( p \), then \( a \) and \( b \) are proportional as points of \( \mathbb{R}^3 \) and equal as points of \( \mathbb{P}^2 \) (this is another example of our abuse of notations).

Our next goal is to relate this notion of polarity to the one discussed in the previous section.

**Lemma.** Suppose that \( a \in S^1 \), i.e., \( Q(a) = 0 \). Then \( a^\perp \) is the tangent to \( S^1 \) at \( a \). In particular, \( a \) is the only point of intersection of \( a^\perp \) and \( S^1 \).

**Proof.** Since \( Q(a) = 0 \), we have \( \langle a, a \rangle = 0 \), and therefore \( a \in a^\perp \). Since \( a \in S^1 \), the point \( a \) is not contained in the line at infinity, and hence has the form \( a = [u : v : 1] \) with \( u^2 + v^2 = 1 \). A point \([x : y : 1] \) is contained in \( a^\perp \) if and only if \( \langle (x, y, 1), (u, v, 1) \rangle = 0 \), i.e., \( xu + yv - 1 = 0 \), or \( xu + yv = 1 \). But the last equation is exactly the equation of the Euclidean tangent to the unit circle \( S^1 \) at the point \((u, v)\). In addition to such points \([x : y : 1] \) the line \( a^\perp \) contains a point at infinity.

**Proposition.** Suppose \( a \) is a point in \( \mathbb{P}^1 \). Then \( a \) lies outside of the unit circle if and only if \( a^\perp \) intersects the unit disc. If \( a \) is outside of the unit circle (so \( a^\perp \) intersects the unit disc), then \( a \) is geometrically polar to the intersection of \( a^\perp \) with the unit disc.

**Proof.** Suppose first that \( a \) lies outside of the unit circle. Let \( t \) and \( t' \) be the two tangents to \( S^1 \) passing through the point \( a \). Let \( b \) and \( b' \) be the corresponding points of tangency. By the lemma, \( t \) is algebraically polar to \( b \) and \( t' \) is algebraically polar to \( b' \). Therefore, \( \langle a, b \rangle = 0 \) and \( \langle a, b' \rangle = 0 \). By the symmetry of the form \( \langle \cdot, \cdot \rangle \), we have \( \langle b, a \rangle = 0 \) and...
\( \langle b', a \rangle = 0 \). In other words, \( b, b' \in a^\perp \). We see that \( a^\perp \) is the line passing through the two points \( b \) and \( b' \), and hence the point \( a \) of intersection of the two tangents \( t \) and \( t' \) to \( S^1 \) at these points is geometrically polar to the intersection of \( a^\perp \) with the unit disc.

Now, suppose \( a \) lies inside of the unit circle (i.e. in the unit disc). If \( a^\perp \) intersects the unit disc, then we can consider the point \( p \) geometrically polar to the intersection of \( a^\perp \) with the unit disc. The point \( p \) lies outside of the unit circle, and therefore \( p \) is geometrically polar to the intersection of \( p^\perp \) with the unit disc by the previous paragraph. It follows that \( p \) is geometrically polar to the intersections of both \( a^\perp \) and \( p^\perp \) with the unit disc. Hence, these intersections are equal, and \( a^\perp = p^\perp \). As we noticed above, this implies that \( a = p \). This contradicts to the fact that \( a \) lies inside of the unit circle, and \( p \) lies outside. The contraction shows that if \( a \) lies inside of the unit circle, then \( a^\perp \) does not intersect the unit disc. This completes the proof.

**Theorem 2.** Let \( l \) and \( m \) be two lines in the Klein model, and let \( a \) and \( b \) be the points geometrically polar to them. The following four conditions are equivalent:

(i) \( l \) is orthogonal to \( m \);
(ii) \( \langle a, b \rangle = 0 \);
(iii) \( b^\perp \ni a \);
(iv) \( a^\perp \ni b \).

**Proof.** By the proposition, \( l \) and \( m \) are the intersections of \( a^\perp \) and \( b^\perp \), respectively, with the unit disc. By Theorem 1, \( l \) is orthogonal to \( m \) if and only if \( b^\perp \) contains \( a \), i.e., if and only if \( \langle a, b \rangle = 0 \). Finally, \( \langle a, b \rangle = 0 \) is equivalent to \( \langle b, a \rangle = 0 \), and therefore to \( a^\perp \ni b \).

Note the usefulness of the symmetry of the form \( \langle \cdot, \cdot \rangle \) in the above arguments.

For a more systematic treatment of polarity and its use in hyperbolic geometry, we recommend E. Rees’s book [16].

**5. THE KLEIN MODEL IN A VECTOR SPACE WITH A PAIRING.** It is now clear that the notions of points and lines in the Klein model and of the orthogonality of such lines can be formulated entirely in terms of the pairing \( \langle \cdot, \cdot \rangle \) and the associated quadratic form \( Q \). Namely, the circle \( Q(p) = 0 \) in \( \mathbf{P}^2 \) divides \( \mathbf{P}^2 \) into two parts; one of them is the unit disc, and the other is a Möbius band. A point \( p \) is contained in the unit disc if and only if \( Q(p) < 0 \). The lines are the intersections of the projective lines with the disc part, and Theorem 2 allows us to define orthogonality in terms of the pairing \( \langle \cdot, \cdot \rangle \).

Let \( V \) be some 3-dimensional vector space endowed with a pairing \( D_V(\cdot, \cdot) \). Suppose that \( D_V \) is symmetric, i.e., \( D_V(p, q) = D_V(q, p) \) for all \( p, q \). As in the previous section, the quadratic form \( Q_V \) associated to the pairing \( D_V \) is defined by the formula \( Q_V(a) = D_V(a, a) \). Let \( W \) be some other 3-dimensional vector space endowed with a symmetric pairing \( D_W(\cdot, \cdot) \), and let \( Q_W \) be the associated quadratic form. We call a vector space isomorphism \( f : V \rightarrow W \) an isometry between \( (V, D_V) \) and \( (W, D_W) \) if \( D_W(f(p), f(q)) = D_V(p, q) \) for all \( p, q \in V \). Usually the pairings are not mentioned explicitly and we speak just about an isometry \( f : V \rightarrow W \).

Suppose that there is an isometry \( f : \mathbf{R}^3 \rightarrow V \), where \( \mathbf{R}^3 \) is endowed with our pairing \( \langle \cdot, \cdot \rangle \). In this case we can define a Klein model using \( V \) and \( D_V(\cdot, \cdot) \) instead of \( \mathbf{R}^3 \) and \( \langle \cdot, \cdot \rangle \). Namely, the role of the circle \( S^1 \) will be played by the conic given by the equation \( Q_V(p) = 0 \), the role of the unit disc will be played the interior of this conic given by the inequality \( Q_V(p) < 0 \), etc. Any theorem about lines and orthogonality in the Klein model can be translated to this model using isometries.
proved for this new model will automatically be true for the old one (one can use \( f \) to go from one model to the other). This trivial observation allows us to replace \( \mathbb{R}^3 \) by a vector space with a richer structure. This freedom of choosing \( V \) and \( D_V \) turns out to be a key tool in our proof of the theorem about the altitudes in hyperbolic geometry.

**Lemma.** If an isomorphism of vector spaces \( f : V \to W \) takes \( Q_V \) into \( Q_W \), i.e., \( Q_W(f(p)) = Q_V(p) \) for all \( p \in V \), then \( f \) is an isometry.

**Proof.** Let \( D = D_V \). Recall the polarization identity

\[
D(p, q) = \frac{1}{2} (D(p + q, p + q) - D(p, p) - D(q, q))
\]

(to check this identity, expand \( D(p + q, p + q) \) by bilinearity and use the symmetry \( D(p, q) = D(q, p) \)). We can rewrite it as follows:

\[
D(p, q) = \frac{1}{2} (Q_V(p + q) - Q_V(p) - Q_V(q)). \tag{1}
\]

Therefore, \( D = D_V \) can be reconstructed from \( Q_V \) (and the vector space structure of \( V \)). Of course, the same applies to \( D_W \) and \( Q_W \). It follows that if an isomorphism \( f \) takes \( Q_V \) into \( Q_W \), then it is an isometry.

Given any function \( Q_V \) on \( V \) that is a homogeneous polynomial of degree 2 when written in some (and therefore in any) coordinates on \( V \), one can use formula (1) in order to define a symmetric pairing. Although we do not need this general fact for our proofs, knowing that this is true allows us to start the search for an appropriate pairing from a quadratic form.

6. ORTHOCENTERS IN THE KLEIN MODEL. The main theorem about the altitudes of triangles in the Klein model is the following.

**Theorem 3.** The three altitudes of a triangle in the Klein model intersect in a point of the projective plane, which is called the orthocenter of the triangle.

Notice that the orthocenter may lie outside of the Klein model itself (i.e., outside of the unit disc). But if two altitudes do intersect in a point in the Klein model, then the theorem implies that the third altitude also passes through this point, and in this case the orthocenter is a point of the Klein model. In a more classical spirit one may say that the triangle has an orthocenter in this case. In the next section we will discuss when this happens.

Using our freedom of choice of a 3-dimensional vector space \( V \) and a pairing on it, in the proof of the theorem we will use the space \( sl(2) \) of real 2-by-2 matrices with trace 0. Every such matrix has the form

\[
\begin{pmatrix}
x & y \\
z & -x
\end{pmatrix},
\]

and we will use \((x, y, z)\) as standard coordinates on \( sl(2) \).

We will choose the pairing \( D = D_V \) in such a way that the associated quadratic form is equal to \(-\det:\)

\[-\det\begin{pmatrix}
x & y \\
z & -x
\end{pmatrix} = -(−x^2 − yz) = x^2 + yz.\]
By the remark at the end of the previous section such a pairing must exist. Anyhow, we will give a simple and explicit formula for it in the next lemma. Note that \( \text{tr}(XY) = \text{tr}(YX) \), and hence the bilinear form \( \frac{1}{2} \text{tr}(XY) \) is symmetric.

**Lemma.** The quadratic form associated to \( D(X, Y) = \frac{1}{2} \text{tr}(XY) \) is equal to \(- \det \). Here \( \text{tr} \) denotes the trace of a matrix.

**Proof.** We need to check that

\[
- \det(X) = \frac{1}{2} \text{tr}(X^2)
\]

for any matrix

\[
X = \begin{pmatrix} x & y \\ z & -x \end{pmatrix}.
\]

Clearly,

\[
X^2 = \begin{pmatrix} x & y \\ z & -x \end{pmatrix} \begin{pmatrix} x & y \\ z & -x \end{pmatrix} = \begin{pmatrix} x^2 + yz & \cdot \\ \cdot & zy + x^2 \end{pmatrix},
\]

where dots denote the terms we are not interested in. It follows that \( \text{tr}(X^2) = 2(y^2 + xz) = -2 \det(X) \).

Let \( f : \mathbb{R}^3 \to \text{sl}(2) \) be the map

\[
f(x, y, z) = \begin{pmatrix} x & y - z \\ y + z & -x \end{pmatrix}.
\]

Clearly, \( f \) is a vector space isomorphism, and

\[- \det(f(x, y, z)) = x^2 + y^2 - z^2 = Q(x, y, z).\]

So, \( f \) takes the quadratic form \( Q \) associated to \( \langle \cdot, \cdot \rangle \) into the quadratic form \(- \det \) associated to \( D(\cdot, \cdot) \). By the lemma from Section 5, \( f \) is an isometry. So by the discussion in Section 5, we can use \( \text{sl}(2) \) and \( D(\cdot, \cdot) \) instead of \( \mathbb{R}^3 \) and \( \langle \cdot, \cdot \rangle \) in our proofs.

The minus sign in front of \( \det \), which may look strange at first sight, is needed for the existence of \( f \) taking \( Q \) to our quadratic form on \( \text{sl}(2) \). This immediately follows from Silvester’s law of inertia of quadratic forms. Of course, we need the existence of \( f \) only for our choice of sign.

The next lemma will allow us to relate commutators and polarity. As usual, for any 2-by-2 matrices \( A \) and \( B \), we write \( [A, B] \) to denote the commutator \( AB - BA \) of \( A \) and \( B \). Using the fact that \( \text{tr}(AB) = \text{tr}(BA) \), we see that \( \text{tr}([A, B]) = \text{tr}(AB - BA) = \text{tr}(AB) - \text{tr}(BA) = 0 \), so \( [A, B] \in \text{sl}(2) \).

**Lemma about Traces.** For any 2-by-2 matrices \( A \) and \( B \), we have \( \text{tr}(A[A, B]) = 0 \).

**Proof.** We will apply the basic property of the trace \( \text{tr}(XY) = \text{tr}(YX) \) with \( X = A \), \( Y = AB \). Namely,

\[
\text{tr}(A[A, B]) = \text{tr}(AAB - AB) = \text{tr}(AAB) - \text{tr}(ABA)
\]

\[
= \text{tr}(ABA) - \text{tr}(ABA) = 0.
\]

This completes the proof.
In order to use this lemma, we need to know when this trace is not equal to zero for trivial reasons, i.e., when \([A, B] \neq 0\). It turns out that this is essentially always the case.

**Lemma.** If \(A, B \in \text{sl}(2)\) and \([A, B] = 0\), then the matrices \(A\) and \(B\) are proportional.

**Proof.** Let
\[
A = \begin{pmatrix} X & Y \\ Z & -X \end{pmatrix}, \quad B = \begin{pmatrix} x & y \\ z & -x \end{pmatrix}.
\]
Then
\[
AB = \begin{pmatrix} Xx + Yz & Xy - Yx \\ Zx - Xz & Zy + Xz \end{pmatrix}, \quad BA = \begin{pmatrix} xX + yZ & xY - yX \\ zX - xZ & zY + xX \end{pmatrix}.
\]
Therefore, if \(AB = BA\), then
\[
Yz = yZ, \quad 2Xy = 2Yx, \\
2Zx = 2Xz, \quad Zy = zY.
\]
So, in this case we have
\[
Y/y = Z/z, \quad X/x = Y/y, \quad X/x = Z/z.
\]
It follows that the matrices \(A\) and \(B\) are proportional.

Following the conventions of Section 4, we will not distinguish between a nonzero matrix \(X \in \text{sl}(2)\) and the point defined by \(X\) in the projective plane corresponding to \(\text{sl}(2)\).

**Corollary.** Suppose that \(A\) and \(B\) represent two different points of \(\mathbf{P}^2\). Then \([A, B] \neq 0\) and the line \([A, B]^\perp\) passes through \(A\) and \(B\).

**Proof.** Since \(A\) and \(B\) represent different points, the matrices \(A\) and \(B\) are not proportional. Therefore, \([A, B] \neq 0\). By the definition of \(D\) and the lemma about traces, \(D(A, [A, B]) = 0\). Similarly,
\[
D(B, [A, B]) = -D(B, [B, A]) = 0.
\]
The corollary follows.

A useful special case of this situation occurs if the line \([A, B]^\perp\) intersects the unit disc. By the proposition from Section 4, in this case \([A, B]\) is the point outside the unit circle that is polar to the intersection of \([A, B]^\perp\) with the unit disc. In particular, this is true if either \(A\) or \(B\) is inside of the unit disc. Indeed, the line \([A, B]^\perp\) contains \(A\) and \(B\) by the corollary.

Now we can describe the altitudes in terms of commutators and polarity.

**Main Lemma.** Let \(A, B, C\) be three points in \(\text{sl}(2)\) representing three vertices of a triangle in the Klein model. Then \([C, [A, B]]\) is nonzero, and therefore defines a point in \(\mathbf{P}^2\). The line polar to this point contains the point defined by \(C\) and the intersection of
this line with the unit disc is orthogonal to the line in the Klein model passing through A and B. In other words, the line polar to \([C, [A, B]]\) is the projective extension of the altitude of the triangle \(ABC\) passing through \(C\) and orthogonal to \(AB\).

**Proof.** First, let us show that \([C, [A, B]]\) is nonzero. Since \(A\) and \(B\) are inside of the unit disc, \([A, B]\) is a point outside the unit circle by the remarks following the last corollary. Since \(C\) is assumed to be inside of the unit disc, \(C \neq [A, B]\), and \([C, [A, B]] \neq 0\) by the last corollary.

Let \(γ = [C, [A, B]]\). By the last corollary, \(C \in γ^⊥\), and, hence, \(C\) is contained in the line polar to \(γ\). Also, by the last corollary, \([A, B] \in γ^⊥\). This means that \(D([A, B], γ) = 0\), and (using the symmetry of our form again!) that \(γ \in [A, B]^⊥\).

Note that \([A, B]\) is a point outside of the unit circle (as we noted in the previous paragraph), and therefore \([A, B]\) is a point geometrically polar to the intersection of \([A, B]^⊥\) with the unit disc. Also, \(C\) is a point inside of the unit circle, and therefore \(γ\) is polar to the intersection of \(γ^⊥ = [C, [A, B]]^⊥\) with the unit disc (by remarks following the last corollary), with \(C\) and \([A, B]\) in the role of \(A\) and \(B\), respectively.

Now we can apply Theorem 2 to \(γ^⊥\) and \([A, B]^⊥\) (more precisely, their intersections with the unit disc) in the role of \(l\) and \(m\) and \(γ\) and \([A, B]\) in the role of \(a\) and \(b\), respectively.

By Theorem 2 \(γ \in [A, B]^⊥\) implies that the lines polar to \(γ\) and \([A, B]\) are orthogonal. Using the last corollary once more, we see that the line polar to \([A, B]\) passes through \(A\) and \(B\).

Collecting all these facts together, we see that the line polar to \(γ\) contains \(C\) and is orthogonal to the line passing through \(A\) and \(B\), as claimed.

The next lemma provides us with a tool for establishing that three lines have a common point. It corresponds to a key theorem in Arnol’d’s approach; see [3, Theorem 4].

**Lemma about Common Points.** If three nonzero matrices \(α, β, γ\) satisfy the relation \(α + β + γ = 0\), then the three projective lines \(α^⊥, β^⊥, γ^⊥\) have a common point.

**Proof.** The linear dependence among \(α, β, γ\) implies that the equations \(D(X, α) = D(X, β) = D(X, γ) = 0\) have a nonzero solution \(X\). Such a solution represents a common point of these lines.

Now we are ready to prove Theorem 3 by using the Jacobi identity

\[
[[[A, B], C] + [[B, C], A] + [[C, A], B] = 0.
\]

This identity is well known and can be verified by simply substituting for all commutators their definition and canceling the terms pairwise.

**Proof of Theorem 3.** Let \(α = [A, [B, C]]\), \(β = [B, [C, A]]\), \(γ = [C, [A, B]]\). By the main lemma, the lines \(α^⊥, β^⊥, γ^⊥\) contain the three altitudes of the triangle \(ABC\). By the Jacobi identity, \(α + β + γ = 0\). Now the lemma about common points implies that these three lines have a common point.

**Remark 1.** There are three possibilities for the orthocenter of a triangle in the Klein model. First, the orthocenter may be contained in the Klein model itself. In this case we have the classical situation, namely, the three altitudes have a common point in the
hyperbolic plane. Second, the orthocenter may be contained in the unit circle (or in the
conic given by $Q_V(p) = 0$ in general). In this case the three altitudes have a common
point at infinity. In other words, the altitudes are asymptotically parallel. Third, the
orthocenter may lie outside of the closed unit disc. In the last case, let us consider
the line $l$ polar to the orthocenter $O$. Since every altitude contains $O$, Theorem 2
implies that every altitude is orthogonal to $l$. Therefore, in this case the altitudes have
a common perpendicular.

By summing up these observations, we see that the altitudes of a hyperbolic tri-
gle either have a common point, or are asymptotically parallel, or have a common
perpendicular.

**Remark 2.** The above arguments are largely independent of the assumption that the
points $A$, $B$, and $C$ are contained in the Klein model (based on $sl(2)$). This makes it
possible to base a proof of a classical theorem of Chasles on these arguments. In order
to state this theorem, we need a definition. Let us say that a point $a$ is polar to a line $l
in the projective plane if $l$ is polar to $a$ in the sense of Section 4, i.e., if $l = a^\perp$. As we
noted right after the definition of algebraically polar lines (see Section 4), $a$ is uniquely
defined by $l$.

**Chasles’s theorem.** Let $A$, $B$, and $C$ be three different points in the projective plane,
and let $a$, $b$, $c$ be the points polar to the lines $BC$, $CA$, $AB$, respectively. Suppose that
$a \neq A$, $b \neq B$, $c \neq C$. Then the three projective lines $aA$, $bB$, $cC$ are (obviously well
defined and) concurrent, i.e., have a common point.

**Proof.** By the corollary, $[A, B]^\perp$ is the line $AB$, i.e., $[A, B]$ is polar to $AB$. This means
that $c = [A, B]$. Similarly, $b = [C, A]$ and $a = [B, C]$. Using the corollary again,
we see that $[c, C]^\perp$ is the line $cC$. This means that the line $cC$ is polar to $[c, C] =
[[A, B], C]$. Similarly, $bB$ is polar to $[[C, A], B]$, and $aA$ is polar to to $[[B, C], A]$. Now the lemma about common points implies, in view of the Jacobi identity, that the
lines $aA$, $bB$, $cC$ have a common point.

In the context of the theorem about the altitudes the assumption $a \neq A$, $b \neq B$,
$c \neq C$ holds automatically, because $A$, $B$, $C$ are contained in the Klein model, and the
points $a$, $b$, $c$ polar to the sides of the triangle $ABC$ are outside of the Klein model.
In this case Theorem 1 implies that $aA$, $bB$, $cC$ are the altitudes of the triangle $ABC$.
By combining this fact with Chasles’s theorem, we can deduce the theorem about the
altitudes.

For a classical approach to Chasles’s theorem, we recommend [6] or [8]; see [6, Theorem 3.22] and [8, Theorem 7.31].

7. WHEN DO ALTITUDES INTERSECT? In this section, we will give a partial
answer to the question of when the altitudes of a hyperbolic triangle intersect (inside
of the hyperbolic plane itself). Namely, we will prove the following.

**Theorem 4.** If all angles in a hyperbolic triangle are $\leq 2\pi/3 = 120^\circ$, then its alti-
tudes have a common point. For every $\gamma > 2\pi/3$ there exists a triangle with an angle
equal to $\gamma$, whose altitudes do not intersect and are not asymptotically parallel.

We will deal with acute triangles first. To this end, we need the following simple
lemma.
Lemma. Consider a hyperbolic triangle $ABC$. If the altitude from $A$ intersects the line $BC$ outside of the segment $BC$, say, on the side of $C$, then the angle $\angle ACB$ is obtuse. If this altitude intersects the segment $BC$, then the angles $\angle ABC$ and $\angle ACB$ are acute.

Proof. Let $H$ be the point of intersection of the altitude from $A$ with the line containing $BC$. See Figure 2. Suppose that $H$ lies outside the segment $BC$ on the side of $C$. If the angle $\angle ACB$ is not obtuse, then the sum of the angles of the triangle $ACH$ is more than $\angle ACH + \angle AHC = \angle ACH + \pi/2 > \pi$, a contradiction. Therefore $\angle ACB$ is obtuse. The case when $H$ lies in the segment $BC$ is similar. 

![Figure 2.](image)

Corollary. The altitudes of an acute triangle have a common point inside of the triangle.

Proof. The first part of the lemma implies that any two altitudes intersect inside of the triangle, exactly as in Euclidean geometry. It remains to apply Theorem 3. 

So, in order to prove the theorem, it remains to consider obtuse triangles.

Proof of Theorem 4 for obtuse triangles. We will use the Klein model from Section 2. We may assume that the angle $\angle BAC$ is obtuse. It follows from the last lemma that the altitudes from $B$ and $C$ intersect the lines $AB$ and $AC$ in some points of the rays starting at $A$ and not containing $C$ and $B$, respectively.

At this point, our main difficulty is that angles in the Klein model are, in general, different from Euclidean angles. We could switch to the Poincaré model, but then we would have to deal with arcs of circles instead of segments. Fortunately, we are interested in a single angle, namely $\angle BAC$. Using the correspondence between the Klein and Poincaré models from Section 2, we see that angles in the Klein model agree with Euclidean angles at the origin (since $V \circ S^{-1}$ takes diameters into themselves).

We now invoke a basic property of the hyperbolic plane: for any two points there is a hyperbolic motion taking one point into the other. So, we can move our triangle $ABC$ into such a position that $A$ coincides with the origin, and therefore the hyperbolic angle $\angle BAC$ is equal to the Euclidean angle.

In this case the sides $AB$ and $AC$ are contained in diameters of the unit circle. The point polar to a diameter lies at infinity. Any projective line passing through this point is parallel (in the Euclidean sense) to the tangents to the circle at the endpoints of the diameter, and, therefore, is orthogonal to the diameter. Hence, Theorem 1 implies that hyperbolic orthogonality of a line to the line $AB$ or the line $AC$ is the same as Euclidean orthogonality. In other words, the hyperbolic altitudes of the triangle $ABC$ from $B$ and $C$ are the same as the Euclidean altitudes. We would like to know when these two altitudes intersect inside of the unit disc.
Now, if for a triangle $ABC$ as above (with the vertex $A$ situated at the origin) these two altitudes intersect inside of the unit disc, then for any triangle $AB'C'$ such that $B'$ and $C'$ are contained in the sides $AB$ and $AC$, respectively, the altitudes from $B'$ and $C'$ also intersect inside of unit disc. See Figure 3. This suggests moving the vertices $B$ and $C$ as far as possible along two rays starting at $A$ and looking at what happens. So, we will move them to infinity and consider an ideal triangle $ABC$ with $A$ at the origin and $B$ and $C$ lying on the unit circle. Clearly, if for such an ideal triangle the altitudes from $B$ and $C$ intersect inside of the unit circle, then for any triangle $AB'C'$ with $B'$ and $C'$ contained in the rays $AB$ and $AC$, respectively, the altitudes from $B'$ and $C'$ also intersect inside of the unit circle. Moreover, if the altitudes from $B$ and $C$ intersect on the unit circle, then altitudes from $B'$ and $C'$ still intersect inside of the unit circle, if at least one of the points $B'$ and $C'$ is inside of the unit circle. On the contrary, if the altitudes from $B$ and $C$ intersect outside of the unit circle, then for some points $B'$ and $C'$ on the rays $AB$ and $AC$ which are sufficiently close in the Euclidean sense to $B$ and $C$ respectively, the altitudes from $B'$ and $C'$ also intersect outside of the unit circle.

Therefore, it is sufficient to prove that for ideal triangles $ABC$ as above the point of intersection of the altitudes from $B$ and $C$ is contained in the (closed) unit disc if the angle $\angle BAC$ is $\leq 2\pi/3$, and is outside of it if this angle is $> 2\pi/3$.

So, let us consider such an ideal triangle $ABC$. See Figure 4. The whole configuration is symmetric with respect to the line bisecting the angle $\angle BAC$. Let $XY$ be the intersection of this line with the (closed) unit disc; we may assume that the ray $AY$ is directed inside the triangle $BAC$. Let $O$ be the intersection of the altitudes from $B$ and $C$. By symmetry, $O$ is contained in the bisecting line, and, in addition, $O$ and $X$ are on the same side of $A$ (since the angle $\angle BAC$ is obtuse). By definition, $CO$ is orthogonal to $AB$; let $D$ be the point of intersection of these two lines. Let $\alpha$ be the angle $\angle CAY$, equal to the angle $\angle YAB$. Clearly, the angle $\angle DAO$ is equal to the angle $\angle YAB$, which is equal to $\alpha$ by symmetry. Let $\beta$ be the Euclidean angle $\angle DOA$.

Suppose that $O$ is contained inside of the unit circle or on the unit circle itself. Then $\beta \geq \angle CXY$, and $\angle CXY$ is equal to half the angle $\angle CAY$. It follows that $\beta \geq \alpha/2$. Now, consider the right Euclidean triangle $ODA$. We see that $\beta + \alpha = \pi/2$. Together with the inequality $\beta \leq \alpha/2$ this implies $\alpha/2 + \alpha \leq \pi/2$, so $3\alpha \leq \pi$ and $2\alpha \leq 2\pi/3$. It remains to notice that $\angle BAC = 2\alpha$.

Suppose now that the orthocenter is outside the unit circle. In this case $\beta < \angle CXY$, and, arguing as in the previous paragraph, we conclude that $\angle BAC = 2\alpha > 2\pi/3$. ■
8. ORTHOCENTERS IN SPHERICAL GEOMETRY. In order to relate the theorem about the altitudes in spherical geometry with the Jacobi identity, we will use the usual vector product of vectors in $\mathbb{R}^3$. The Jacobi identity for the vector product has the well-known form

$$(a \times b) \times c + (b \times c) \times a + (c \times a) \times b = 0.$$

Let $S^2$ be the unit sphere in $\mathbb{R}^3$. The points in spherical geometry are the points of $S^2$, i.e., the unit vectors in $\mathbb{R}^3$. The lines in spherical geometry are the intersections of planes (here we use the term planes in the sense of linear algebra, i.e., planes are 2-dimensional vector subspaces of $\mathbb{R}^3$) with $S^2$, and the angle between two spherical lines is equal to the angle between the corresponding planes, which is, in turn, equal to the angle between the lines orthogonal to these planes.

Let $abc$ be a spherical triangle. As usual, the vertices $a$, $b$, $c$ are assumed to be distinct. In addition, we will assume that these vertices are different from the points $-a$, $-b$, $-c$. In this case the sides $ab$, $bc$, and $ca$ are well defined. Notice that in this case the vector products $a \times b$, $b \times c$, and $c \times a$ are nonzero.

If $c$ is orthogonal to the plane spanned by $a$ and $b$, then every spherical line passing through $c$ is orthogonal to $ab$. Otherwise, only one spherical line passing through $c$ is orthogonal to $ab$. So, we will assume that none of the vectors $a$, $b$, $c$ is orthogonal to the plane spanned by the other two. In this case the three altitudes of $abc$ are well defined, and the vector products $(a \times b) \times c$, $(b \times c) \times a$, and $(c \times a) \times b$ are nonzero.

By definition, the vector product $a \times b$ is orthogonal to the plane (i.e., vector subspace) $P$ spanned by $a$ and $b$. The vector product $(a \times b) \times c$ is orthogonal to $a \times b$, and therefore is contained in $P$. Now, consider the plane $\gamma$ orthogonal to $(a \times b) \times c$.

It follows that the intersection of $\gamma$ with the unit sphere is the spherical line passing through $c$ and orthogonal to the spherical line $ab$ (which is equal to the intersection of $P$ with $S^2$). In other words, this intersection is the altitude from $c$ of the spherical triangle $abc$. Similarly, the intersections of the planes $\alpha$ and $\beta$ orthogonal to $(b \times c) \times a$ and $(c \times a) \times b$ respectively are the altitudes from $a$ and $b$.

Since the vectors $(a \times b) \times c$, $(b \times c) \times a$, $(c \times a) \times b$ are linearly dependent by the Jacobi identity, either the intersection of the three planes $\gamma$, $\alpha$, $\beta$ is a line, or
these three planes coincide. In the first case all three altitudes of the triangle \( abc \) pass through the two points in \( S^2 \) corresponding to the intersection line. (The altitudes have two common points, since two different lines in spherical geometry always intersect at two points.) In the second case all three vertices \( a, b, c \) are contained in the plane \( \alpha = \beta = \gamma \). Clearly, in this case the altitudes intersect in two unit vectors orthogonal to this plane.

The argument in the previous paragraph is similar to the use of the lemma about common points in Section 6.

This completes our discussion of the altitudes in spherical geometry. Further applications of the Jacobi identity to spherical geometry in the spirit of Arnol’d’s ideas are discussed in [17].

9. ORTHOCENTERS IN EUCLIDEAN GEOMETRY: KIRILLOV’S PROOF.

In this section we present the proof of the Euclidean altitudes theorem outlined by A. A. Kirillov in [12, Appendix III, §1, Exercise 1]. This proof is very close in spirit and in outline to our proofs of the altitudes theorems in hyperbolic and spherical geometry.

We will denote by \((u, v)\) the usual scalar product of two vectors \( u \) and \( v \) in \( \mathbb{R}^3 \).

Let \( P \subset \mathbb{R}^3 \) be the plane consisting of all points \((1, x_1, x_2)\), where \( x_1, x_2 \in \mathbb{R} \). Our triangles will be contained in \( P \), but the ambient space \( \mathbb{R}^3 \) will play a crucial role in the proof.

Let \( e_0 = (1, 0, 0) \). If a vector \( u \in \mathbb{R}^3 \) is not proportional to \( e_0 \), then the intersection of the plane orthogonal to \( u \) with \( P \) is a line in \( P \), which we will denote by \( u^\perp \).

For a vector \( u \), let \( \overline{u} = (u, e_0)e_0 \), and \( \widetilde{u} = u - \overline{u} \). If \( u = (u_0, u_1, u_2) \), then \( \overline{u} = (u_0, 0, 0) \) and \( \widetilde{u} = (0, u_1, u_2) \).

Proposition.

(i) If \( u = (u_0, u_1, u_2) \), then the line \( u^\perp \) is given by the equation \( u_0 + u_1x_1 + u_2x_2 = 0 \).

(ii) If \( a \) and \( b \) are two different points in \( P \), then the line \((a \times b)^\perp \) contains \( a \) and \( b \).

(iii) Two lines \( u^\perp \) and \( v^\perp \) are orthogonal if and only if the vectors \( \widetilde{u} \) and \( \widetilde{v} \) are orthogonal.

(iv) If \( a \in P \), and \( u^\perp \) is a line in \( P \), then the line \((a \times \widetilde{u})^\perp \) contains \( a \) and is orthogonal to \( u^\perp \).

Proof. (i) is clear. (ii) follows from the fact that the vectors \( a \) and \( b \) are orthogonal to \( a \times b \). (iii) follows from (i) and the fact that \( \widetilde{u} = (0, u_1, u_2) \), if \( u = (u_0, u_1, u_2) \).

Let us prove (iv). Since \( a \) is orthogonal to \((a \times \widetilde{u})\), it is contained in \((a \times \widetilde{u})^\perp \). By (iii), \((a \times \widetilde{u})^\perp \) is orthogonal to \( u^\perp \) if and only if \((a \times \widetilde{u}) - (a \times \overline{u})\) is orthogonal to \( \widetilde{u} \). It remains to notice that both vectors \((a \times \widetilde{u})\) and \((a \times \overline{u})\) are orthogonal to \( \overline{u} \); the first one by the main property of the vector product (which we have already used many times), and the second one because \( \overline{u} \) is orthogonal to \( \overline{w} \) for any two vectors \( v \) and \( w \). This completes the proof.

Lemma about Common Points. Suppose that none of the three vectors \( u, v, w \) is proportional to \( e_0 \). If \( u + v + w = 0 \), then the lines \( u^\perp, v^\perp, w^\perp \) either have a common point or are parallel.

Proof. Since the vectors \( u, v, w \) are linearly dependent, the planes orthogonal to them are either equal or intersect along a line \( l \). If these planes are equal, then our lines
are also equal (and so are parallel). If \( l \) intersects \( P \), then the intersection point is a common point of the lines \( u^{\perp}, v^{\perp}, w^{\perp} \). If \( l \) is parallel to \( P \), then these lines are parallel (by elementary geometry).

Let \( x, y, z \) be the vertices of a triangle in \( P \). By statement (ii) of the last proposition, the line \( (y \times z)^{\perp} \) passes through vertices \( y \) and \( z \). By statement (iv), the line \( (x \times (y \times z))^{\perp} \) contains \( x \) and is orthogonal to the line through \( y \) and \( z \). In other words, this line is the altitude from \( x \) in our triangle \( xyz \). The other two altitudes can be described in a similar way. By the lemma about common points, in order to prove the theorem about the altitudes it is sufficient to prove that (notice that altitudes cannot be parallel)

\[
x \times (y \times z) + y \times (z \times x) + z \times (x \times y) = 0.
\]

By the Jacobi identity,

\[
x \times (y \times z) + y \times (z \times x) + z \times (x \times y) = 0.
\]

Since \( \tilde{u} = u - \overline{u} \), we see that it is sufficient to prove that

\[
x \times (y \times z) + y \times (z \times x) + z \times (x \times y) = 0.
\]

Recalling that \( \overline{u} = (u, e_0) e_0 \), we see that the last identity is equivalent to

\[
(y \times z, e_0) x \times e_0 + (z \times x, e_0) y \times e_0 + (x \times y, e_0) z \times e_0 = 0.
\]

This immediately follows from the lemma below (take \( a = b = e_0 \), and recall that \( a \times a = 0 \) for any \( a \)). So, the theorem about the altitudes is proved modulo the following Lemma.

**Lemma.** Let \( x, y, z, a, b \) be vectors in \( \mathbb{R}^3 \). Then

\[
(y \times z, a) x \times b + (z \times x, a) y \times b + (x \times y, a) z \times b = (x \times y, z) a \times b.
\]

**Proof.** It is sufficient to prove that for any \( c \in \mathbb{R}^3 \) the scalar products of both sides with \( c \) are equal, i.e., that

\[
(y \times z, a) (x \times b, c) + (z \times x, a) (y \times b, c) + (x \times y, a) (z \times b, c) = (x \times y, z) (a \times b, c).
\]

Consider the left-hand side as a function \( D(x, y, z) \) of the triple \( (x, y, z) \). Clearly, \( D(x, y, z) \) is multilinear in \( x, y, z \). The first term is skew-symmetric in \( y \) and \( z \) (i.e., changes sign if \( y \) and \( z \) are interchanged), as is the sum of other two terms. Therefore \( D \) is skew-symmetric in \( y \) and \( z \). Similarly, \( D \) is skew-symmetric in the two other pairs of our variables. By the well-known characterization of the determinant, \( D \) is proportional to the determinant of the matrix with rows \( x, y, z \). The latter is equal to \( (x \times y, z) \), as is also well known. In order to find the coefficient of proportionality, it is sufficient to compute the left-hand side for a particular choice of linearly independent \( x, y, z \). Let us take \( x = (1, 0, 0), y = (0, 1, 0), z = (0, 0, 1) \). Then \( (x \times y, z) = 1 \). Let \( a = (a_0, a_1, a_2), b = (b_0, b_1, b_2), c = (c_0, c_1, c_2) \). Then \( x \times b = (0, -b_2, b_1) \) and

\[
(x \times b, c) = -b_2 c_1 + b_1 c_2 = \det \begin{pmatrix} b_1 & b_2 \\ c_1 & c_2 \end{pmatrix}.
\]
Also, \( y \times z = (1, 0, 0) \) and \( (y \times z, a) = a_0 \). So, the first term on the left-hand side is equal to
\[
a_0 \det \begin{pmatrix} b_1 & b_2 \\ c_1 & c_2 \end{pmatrix}.
\]
The two other terms can be computed in the same way. We see that the left-hand side is equal to the expansion of the determinant
\[
\det \begin{pmatrix} a_0 & a_1 & a_2 \\ b_0 & b_1 & b_2 \\ c_0 & c_1 & c_2 \end{pmatrix}
\]
along the first row. So, for \( x = (1, 0, 0), y = (0, 1, 0), z = (0, 0, 1) \) the left-hand side is equal to this determinant, i.e., to \( (a \times b, c) \). Therefore, the coefficient of proportionality is equal to \( (a \times b, c) \), and in general the left-hand side is equal to \( (x \times y, z) (a \times b, c) \). This proves the lemma.

10. ORTHOCENTERS IN EUCLIDEAN GEOMETRY: OTHER PROOFS. It turns out that the Euclidean altitudes theorem can be deduced from the hyperbolic one.

Another proof of the Euclidean altitudes theorem. There is a situation in which orthogonality in the Klein model is the same as Euclidean orthogonality. Namely, this is the case when one of the lines passes through the center \( 0 \) of the unit disc. In fact, we already encountered this phenomenon in the proof of Theorem 4. We will recall the argument for the convenience of the reader. Let \( l \) be a line in the Klein model (the classical one, see Sections 2 and 3) passing through \( 0 \). Then the point geometrically polar to \( l \) is equal to the point at infinity \( [k] \) of \( k \), where \( k \) is any (Euclidean) perpendicular to \( l \). By Theorem 1 (see Section 3) a line \( m \) is hyperbolically orthogonal to \( l \) if and only if its extension \( \overline{m} \) to a projective line contains \( [k] \), i.e., if it is orthogonal to \( l \) in Euclidean sense. So, two notions of orthogonality agree for such an \( l \).

Let us consider a triangle \( ABC \) in the Klein model having \( 0 \) as one of its vertices, say \( A = 0 \). Then the sides \( AB \) and \( AC \) pass through \( 0 \). By the previous paragraph, the hyperbolic altitudes from \( B \) and \( C \) are also the Euclidean altitudes from \( B \) and \( C \). By the same token, the hyperbolic altitude from \( A = 0 \) is also the Euclidean altitude from \( A \). It follows that the hyperbolic and Euclidean altitudes theorems for such a triangle are equivalent.

Since every Euclidean triangle is similar to a triangle contained in the unit disc and having \( 0 \) as one of its vertices, we see that the hyperbolic altitudes theorem implies the Euclidean one.

One can also deduce the Euclidean altitudes theorem from the spherical one in a similar way. Given a Euclidean triangle, one may put it on a plane tangent to the unit sphere in \( S^2 \) in such a way that one of its vertices is equal to the tangency point. For such a triangle \( ABC \), the Euclidean theorem is equivalent to the spherical theorem for the spherical triangle obtained by radial projection of \( ABC \). We leave the details to the interested readers.

One more proof of the Euclidean altitudes theorem. Let us denote by \( (u, v) \) the usual scalar product of two vectors \( u \) and \( v \) in \( \mathbb{R}^2 \). Let \( abc \), where \( a, b, \) and \( c \) are three vectors
in $\mathbb{R}^2$, be our triangle. We assume that the triangle is nondegenerate. This is equivalent to linear independence of the vectors $b - c$ and $c - a$, say. The linear subspace orthogonal to $b - c$ is given by the equation $(b - c, x) = 0$. The line orthogonal to $b - c$ and passing through $a$ is given by the equation $(b - c, x) = (b - c, a)$. It is the altitude of the triangle $abc$ from the vertex $a$. For the three altitudes we get the following three equations.

\[
\begin{align*}
(b - c, x) &= (b - c, a) \\
(c - a, x) &= (c - a, b) \\
(a - b, x) &= (a - b, c)
\end{align*}
\]

The linear map assigning to $x \in \mathbb{R}^2$ the vector whose components are the left-hand sides of these equations has rank 2 (because our triangle is nondegenerate) and its image consists of all vectors $(A, B, C)$ such that $A + B + C = 0$. The vector whose components are the right-hand sides satisfies this condition, and, therefore, our system of linear equations has a solution $x$. This solution is the point of intersection of the three altitudes. This proves the theorem.

Notice that a key point of this proof is based on an argument similar to the lemma about common points from Section 6.

In [1] Arnol’d claimed that the Jacobi identity “forces the heights of a triangle to cross at one point”. In [2], he claimed more specifically that the Jacobi identity for the ordinary vector product “expresses the altitude theorem” in Euclidean geometry.

As we just saw, the altitude theorem in Euclidean geometry can be deduced from either the hyperbolic or the spherical theorem, which in their turn can be deduced from the Jacobi identity. But both of these proofs require moving the triangle into a special position and singling out one of its vertices. So, these proofs proceed by destroying the intrinsic symmetry of the problem, and then by transferring the problem either to hyperbolic or to spherical geometry.

Kirillov’s proof, presented in Section 9, comes much closer to a justification of the above claims by Arnol’d. Still, it is relatively complicated, and is based on a fairly complicated identity, the last lemma of Section 9, in addition to the Jacobi identity.

In contrast with these proof, our last proof keeps the symmetry of the problem, is carried out entirely in Euclidean geometry (as is Kirillov’s proof), is much simpler, and is somewhat parallel to our proof of the hyperbolic theorem (which also keeps the symmetry of the problem). But it does not use the Jacobi identity. Another proof, similar in spirit, but less elementary and more complicated, can be found in [15, Section 10.3]. Similarly, the classical proofs of this theorem, such as the proofs of Euler and Gauss mentioned in the Introduction, do not use the Jacobi identity.

Given this, it seems that the claim that the Jacobi identity “forces the heights of a triangle to cross at one point” is well justified, but the claim that the Jacobi identity “expresses” the Euclidean theorem is somewhat exaggerated. Rather, the Jacobi identity for the ordinary vector product “expresses” the spherical altitudes theorem, as Section 8 shows. Since Arnol’d claimed in [2] (see p. 30 of the English translation) that the altitudes theorem fails in both hyperbolic and spherical geometries, one may think that he mistook a version of the arguments in Section 8 for a proof of the altitudes theorem in Euclidean geometry. One may guess that this mistake eventually led to his proof of the altitudes theorem in hyperbolic geometry.

Of course, the reader may form a different opinion on these matters.
11. FENCHEL'S THEORY OF LINES AND ORTHOCENTERS. In this section we present another approach to the hyperbolic altitudes theorem. This approach is based on Fenchel’s theory of lines (see [9, Chapter V]). It is algebraically strikingly similar to our proof from Section 6 (and is also based ultimately on the Jacobi identity), but the two proofs are very different in what geometric tools they use. In the next section we will compare the two approaches. Recall that in this section and in the next one we expect much more than before from the reader; namely, the reader should be familiar with the upper half-space model of hyperbolic 3-space and with its group of motions.

Following Fenchel [9], we will work with the upper half-space model $H = \mathbb{R}^3 \times \mathbb{R}_{>0}$ (where $\mathbb{R}_{>0}$ is the set of positive real numbers) of 3-dimensional hyperbolic space. It is convenient to identify $H$ with $\mathbb{C} \times \mathbb{R}_{>0}$. As is well known, this identification leads to the identification of the group of orientation-preserving motions of $H$ with the group of Möbius transformations of $\mathbb{C}$, i.e., with the group of maps $F : \mathbb{C} \cup \{\infty\} \rightarrow \mathbb{C} \cup \{\infty\}$ of the form

$$F(z) = \frac{az + b}{cz + d}$$

such that $ad - bc \neq 0$. In turn, this leads to the identification of the group of motions of $H$ with the group $PGL_2(\mathbb{C})$.

The lines in the upper half-space model are the intersections of the open half-space $\mathbb{C} \times \mathbb{R}_{>0}$ with the circles orthogonal to $\mathbb{C} \times \{0\}$ (note that such circles always have center in $\mathbb{C} \times \{0\}$) and with the lines orthogonal to $\mathbb{C} \times \{0\}$. The lines of the first type are (open) Euclidean semicircles having two endpoints in $\mathbb{C} \times \{0\}$. The lines of the second type are (open) Euclidean half-lines having an endpoint in $\mathbb{C} \times \{0\}$. We think of such a half-line as a semicircle of infinite radius whose other endpoint is $\infty$. In particular, every line can be thought as having two endpoints in $(\mathbb{C} \times \{0\}) \cup \{\infty\}$. In the future, we will identify $\mathbb{C} \times \{0\}$ with $\mathbb{C}$ and $(\mathbb{C} \times \{0\}) \cup \{\infty\}$ with $\mathbb{C} \cup \{\infty\}$.

In [9, Chapter V], Fenchel suggested representing the lines in $H$ by motions of $H$, namely representing a line $l$ by the rotation by the angle $\pi$ around $l$. Since the group of motions is isomorphic to $PGL_2(\mathbb{C})$, this also allows us to represent lines by complex 2-by-2 matrices. Fenchel proved that a matrix $A \in GL_2(\mathbb{C})$ represents a rotation by the angle $\pi$ around a line if and only if $\text{tr} A = 0$.

Following Fenchel, we call nonzero matrices with trace zero line matrices. If a line matrix $A$ is nondegenerate (i.e., $\det A \neq 0$), then it represents a line. Its endpoints in $\hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ are the fixed points of the corresponding Möbius transformation. If a line matrix $A$ is degenerate, then the corresponding Möbius transformation has exactly one fixed point. In this case we say that $A$ represents a degenerate line having $z$ as both of its endpoints, where $z \in \hat{\mathbb{C}}$ is the unique fixed point of the corresponding Möbius transformation. An ordinary line has two different endpoints in $\hat{\mathbb{C}}$. We will denote the line corresponding to a line matrix $A$ by $l_A$, including the degenerate case.

We will say that a degenerate line $l'$ is orthogonal to the ordinary line $l'$ if the only endpoint of $l$ is equal to one of the endpoints of $l'$. As is well known, two ordinary lines in $H$ either have a nondegenerate common perpendicular or are asymptotically parallel (i.e., have a common endpoint). So, with our definitions, two different lines (ordinary or degenerate) always have a unique common perpendicular.

The basic fact relating line matrices with orthogonality is the following.

**Proposition.** Let $A$ and $B$ be two line matrices. If they are both nondegenerate, then $\text{tr}(AB) = 0$ if and only if the lines $l_A$ and $l_B$ intersect and are orthogonal at the inter-
section point. If $A$ is nondegenerate and $B$ is degenerate, then \( \text{tr}(AB) = 0 \) if and only if the (only) endpoint of $l_B$ is also an endpoint of $l_A$. In particular, in both cases $l_A$ is orthogonal to $l_B$.

See [9, Section V.1] for a proof of a more general result. The following corollary and lemma are also special cases of some propositions from that section. The corollary (more precisely, its more general version from [9]) apparently appeared for the first time in the famous paper by Jørgensen [11]; see [11, Section 4]. An exposition of this theory, closer in spirit to Jørgensen [11] than to the more elementary approach of Fenchel [9], is given in a recent book by Marden; see [13, Chapter 7].

**Corollary.** If $A$ and $B$ are line matrices representing two different nondegenerate lines, then $l_{[A,B]}$ is a common perpendicular to $l_A$ and $l_B$.

**Proof.** Since the matrices $A$ and $B$ represent different lines, they are not proportional. By the lemma about zero commutators (see Section 6), this implies that $[A, B] \neq 0$ and, therefore, $l_{[A,B]}$ is well defined. By the lemma about traces (see Section 6) \( \text{tr}(A[A, B]) = 0 \) and \( \text{tr}(B[A, B]) = -\text{tr}(B[B, A]) = 0 \). By the proposition, this implies that both $l_A$ and $l_B$ are orthogonal to $l_{[A,B]}$. \hfill \( \blacksquare \)

In the case when $[A, B]$ is degenerate, the corollary implies that $l_A$ and $l_B$ have a common endpoint, namely, the unique endpoint of the degenerate line $l_{[A,B]}$.

**Lemma about Common Perpendiculars.** If three line matrices $\alpha, \beta, \gamma$ satisfy the relation $\alpha + \beta + \gamma = 0$, then the three projective lines $l_\alpha, l_\beta, l_\gamma$ have a common perpendicular.

**Proof.** The linear dependence among $\alpha, \beta, \gamma$ implies that the equations $\text{tr}(X\alpha) = \text{tr}(X\beta) = \text{tr}(X\gamma) = 0$ have a nonzero solution $X$ such that $\text{tr}(X) = 0$. By the proposition, for such a solution $X$ the line $l_X$ is a common perpendicular to our lines. \hfill \( \blacksquare \)

**Theorem.** Let $abc$ be a triangle in $H$. Its three altitudes are either concurrent (i.e., have a common point in $H$), or have a common (nondegenerate) perpendicular contained in the plane of the triangle $abc$, or are asymptotically parallel (i.e., have a common endpoint).

**Proof.** Let $P$ be the plane of the triangle $abc$ (as usual, we assume that our triangle is nondegenerate, i.e., the points $a$, $b$, $c$ are not contained in a line and, therefore, determine a plane). Let $A$, $B$, $C$ be the line matrices representing the lines orthogonal to $P$ at the points $a$, $b$, $c$, respectively. Then $l_{[A,B]}$ is a common perpendicular to $l_A$ and $l_B$. On the other hand, the line $ab$ is a common perpendicular to $l_A$ and $l_B$ by the choice of $A$ and $B$. Since two lines may have no more than one common perpendicular, this implies that $l_{[A,B]} = ab$. Similarly, $l_{[B,C]} = bc$ and $l_{[C,A]} = ca$.

Next, $l_{[[A,B],[C]]}$ is a common perpendicular to $l_{[A,B]} = ab$ and $l_C$. But the altitude of the triangle $abc$ passing through $c$ and orthogonal to $ab$ is also a common perpendicular to $ab$ and $l_C$. It follows that $l_{[[A,B],[C]]}$ is equal to this altitude. Similarly, $l_{[[B,C],[A]]}$ and $l_{[[C,A],[B]]}$ are the other two altitudes of the triangle $abc$. 

Now, the Jacobi identity
\[
[[A, B], C] + [[B, C], A] + [[C, A], B] = 0
\]
together with the lemma about common perpendiculares implies that the three altitudes of \(abc\) have a common perpendicular.

Let us consider two of the altitudes of \(abc\). Suppose that they intersect at a point \(x\). Clearly, \(x \in P\). In this case the only common perpendicular to these two altitudes is the line orthogonal to the plane \(P\) and passing through \(x\). Since all three altitudes have a common perpendicular, this line is also orthogonal to the third altitude. In particular, the third altitude intersects it. But this line has only one common point with \(P\), namely, \(x\), and the third altitude is contained in \(P\). It follows that \(x\) is a common point of all three altitudes. This proves the theorem in this case.

Suppose that the two altitudes have a common (nondegenerate) perpendicular in \(P\). Since two lines may have no more than one common perpendicular, and we know that the three altitudes have a common perpendicular, this common perpendicular is actually a common perpendicular of all three altitudes. This proves the theorem in this case.

Suppose now that the two altitudes are asymptotically parallel. Let \(x\) be their common endpoint. In this case the common perpendicular is the degenerate line connecting \(x\) to itself. This degenerate line has to be a perpendicular to the third altitude also. This means that the three altitudes share a common endpoint, namely, \(x\). This proves the theorem in this case, and completes the proof of the theorem.

12. TWO WAYS TO ASSIGN LINES TO MATRICES. Let \(A\) be a real line matrix such that \(-\det(A) > 0\). There are two ways to assign a line in a hyperbolic plane to \(A\): assign the line \(l_A\) as in Section 11, or assign the line \(A^\perp\) in the Klein model based on \(\mathfrak{sl}(2)\) as described in the proposition in Section 4. It turns out that these two ways are essentially the same.

In order to make sense out of this claim, we need a way to compare our two models of hyperbolic geometry. Our strategy will be to compare lines by comparing their endpoints, so we have to pay special attention to what happens at infinity.

First, let us consider the hyperbolic plane \(\mathbb{R} \times \mathbb{R}_{>0} \subset \mathbb{C} \times \mathbb{R}_{>0}\) in the upper half-space model. This plane is nothing else than the upper half-plane model of the hyperbolic plane. Its circle at infinity is \((\mathbb{R} \times \{0\}) \cup \{\infty\}\), which we will identify with \(\hat{\mathbb{R}} = \mathbb{R} \cup \{\infty\}\). The standard way to identify the upper half-plane model with the Poincaré unit disc model in \(\mathbb{R}^2\) is to apply to it the inversion with center \((0, -1)\) and radius \(\sqrt{2}\). See, for example, [14, Appendix, p. 364]. This inversion takes \(\hat{\mathbb{R}}\) to the unit circle; the restriction of this inversion to \(\hat{\mathbb{R}}\) is the stereographic projection from \(\hat{\mathbb{R}}\) to the unit circle. As is well known (and easily checked), it takes \(t \in \hat{\mathbb{R}}\) to

\[
\left(\frac{2t}{1 + t^2}, \frac{1 - t^2}{1 + t^2}\right)
\]

(it takes \(\infty\) to \((0, -1)\)). The standard identification of the Poincaré unit disc model with the Klein model (see Section 2) is equal to the identity on the circle at infinity. So, the point at infinity in the Klein model corresponding to \(t \in \hat{\mathbb{R}}\) is still the same point \((0, -1)\). Considered as a point of the projective plane, it is equal to

\([2t : 1 - t^2 : 1 + t^2]\).
if \( t \neq \infty \), and to \([0, -1, 1]\) if \( t = \infty \). In order to pass to the Klein model based on \( sl(2) \), we have to use the map \( f : \mathbb{R}^3 \to sl(2) \) from Section 6. Recall that

\[
f(x, y, z) = \begin{pmatrix} x & y - z \\ y + z & -x \end{pmatrix}.
\]

In particular,

\[
f(2t, 1 - t^2, 1 + t^2) = \begin{pmatrix} 2t & - 2t^2 \\ 2 & -2t \end{pmatrix},
\]

and

\[
f(0, -1, 1) = \begin{pmatrix} 0 & -2 \\ 0 & 0 \end{pmatrix}.
\]

Now, consider a real line matrix

\[
A = \begin{pmatrix} a & b \\ c & -a \end{pmatrix}
\]

such that \(- \det(A) = a^2 + bc > 0\). The endpoints of the line \( l_A \) are the fixed points of the corresponding Möbius transformations of \( \hat{C} \), i.e., they are solutions \( t \) of the equation

\[
\frac{at + b}{ct - a} = t. \tag{3}
\]

Equation (3) is equivalent* to

\[
ct^2 - 2at - b = 0 \tag{4}
\]

for \( t \neq \infty \). Clearly, \( t = \infty \) is a solution of (3) if and only if \( c = 0 \). In this case (4) has only one solution in \( \mathbb{R} \), and we will consider \( t = \infty \) as the second solution in \( \hat{R} \). (Since \( a^2 + bc > 0 \), it cannot happen that \( c = a = 0 \).) If \( A \) is a real line matrix and \(- \det(A) = a^2 + bc > 0\), then both solutions of (4) belong to \( \hat{R} \). This implies that both endpoints of \( l_A \) are contained in \( \hat{R} \), and therefore \( l_A \) is contained in the plane \( \mathbb{R} \times \mathbb{R}_{>0} \).

The line \( A^t \) corresponding to \( A \) in the Klein model based on \( sl(2) \) is given by the equation \( \text{tr}(PA) = 0 \). The point \( t \in \mathbb{R} \) corresponds to an endpoint of this line if and only if

\[
\text{tr}(f(2t, 1 - t^2, 1 + t^2)A) = \text{tr}(\begin{pmatrix} 2t & -2t^2 \\ 2 & -2t \end{pmatrix} \begin{pmatrix} a & b \\ c & -a \end{pmatrix}) = 0,
\]

i.e., if and only if

\[
2at - 2ct^2 + 2b + 2at = 0.
\]

The last equation is equivalent to

\[
ct^2 - 2at - b = 0. \tag{5}
\]

*Notice that if we have \( ct - a = 0 \) for a solution of (4), then \( at + b = ct^2 - at = t(ct - a) = 0 \). But if \( ct - a = at + b = 0 \), then either \( c = a = 0 \), or \( a = b = 0 \), or \( t = a/c = -b/a \), and in all these cases \( a^2 + bc = 0 \), contradicting our assumption that \( a^2 + bc > 0 \).
The point \( t = \infty \) corresponds to an endpoint of \( A^\perp \) if and only if
\[
\text{tr}(f(0, -1, 1)A) = \text{tr}\left( \begin{pmatrix} 0 & -2 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} a & b \\ c & -a \end{pmatrix} \right) = -2c = 0,
\]
i.e., if and only if \( c = 0 \).

The equation (4) describes the endpoints of \( l_A \) (with our convention that \( t = \infty \) is a solution for \( c = 0 \)). The equation (5) describes the points \( t \in \mathbb{R} \) corresponding to the endpoints of \( A^\perp \) under our identification of the two models. If we treat the equation (5) in the same way as (4), i.e., if we consider \( t = \infty \) as a solution of (5) if \( c = 0 \), then (5) covers the case \( t = \infty \) too. Since the two equations (4) and (5) are the same, we see \( l_A \) has exactly the same endpoints as \( A^\perp \), and therefore \( l_A = A^\perp \).

We see that the two ways to assign a line to a matrix agree. This agreement can be extended to an “isomorphism” between the two proofs of the theorem about altitudes, the proof from Section 11 and the proof from Section 6. We leave this task to the interested readers. The two proofs are exactly the same on the algebraic level, but are noticeably different in their geometric form.

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A Special Continued Fraction for the Golden Mean

\[
3 + \frac{-4}{3} = \frac{5}{3}, \quad 3 + \frac{-4 + \frac{5}{3}}{3 + \frac{-4}{3}} = \frac{8}{5}, \quad 3 + \frac{-4 + \frac{5}{3}}{3 + \frac{-4 + \frac{5}{3}}{3 + \frac{-4}{3}}} = \frac{13}{8}, \ldots,
\]

and therefore

\[
3 + \frac{-4}{3} = \phi = \frac{1 + \sqrt{5}}{2}.
\]

—Submitted by Domingo Gomez Morin,
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