

A Converse to the Second Whitehead Lemma

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Introduction

The classical First and Second Whitehead Lemmata state that the first, respectively second, cohomology group of a finite-dimensional semisimple Lie algebra with coefficients in any finite-dimensional module vanishes. It is natural to ask whether the converse is true. For the first cohomology this is well-known:

Theorem 0.1. (A converse to the First Whitehead Lemma). *Any finite-dimensional Lie algebra over the field of characteristic zero such that its first cohomology with coefficients in any finite-dimensional module vanishes, is semisimple.*

Proof. (see, e.g., [7, Chapter 1, Theorem 3.3]). Due to cohomological interpretation of modules extension, the vanishing of cohomology with coefficients in any finite-dimensional module is equivalent to the complete reducibility of any finite-dimensional module. Assuming that a Lie algebra L with such condition has a nonzero abelian ideal I and considering the adjoint representation of L , we get that I is a direct summand of L . But then I also should satisfy the same condition, a contradiction. ■

What about the second cohomology?

Somewhat surprisingly, this question seemingly was not addressed in the literature. The aim of the present elementary note is to show that, essentially, the converse is true too – finite dimensional Lie algebras over a field of characteristic zero such that their second cohomology with coefficients in any finite-dimensional module vanishes, are very close to semisimple ones. In fact, the proof is only a bit more complicated than the proof of Theorem 0.1 and readily follows from results already established in the literature.

One may look at this question from a somewhat different angle. Recently, an interesting class of so-called strongly rigid Lie algebras was investigated in [1] – namely, Lie algebras whose universal enveloping algebra is rigid. It turns out that for such algebras, the second cohomology in the trivial module and in the adjoint module vanishes. Considering these two modules as the most “natural” ones, one may wonder for which Lie algebras the stronger condition – vanishing of the second cohomology with coefficient in any finite-dimensional module – would hold.

What happens in the positive characteristic? As was shown independently by Dzhumadil'daev [5] and Farnsteiner – Strade [6], for any finite-dimensional Lie algebra over a field of positive characteristic and any degree not greater than the dimension of the algebra, there is a module with non-vanishing cohomology in

that degree. (In fact, the low degree cases interesting for us here were settled even earlier – for the first degree cohomology by Jacobson [9, Chap. VI, §3, Theorem 2] and for the second degree cohomology – again by Dzhumadildaev [3]). So the answer is trivially void in this case – one of the rare cases when the answer (but not the proof!) in the positive characteristic turns out to be simpler. The reason for this is the possibility to construct analogues of induced modules with desired cohomological properties by means of various truncated finite-dimensional versions of the universal enveloping algebra. In characteristic zero, such modules would be infinite-dimensional.

Theorem 0.2. (A converse to the Second Whitehead Lemma). *A finite-dimensional Lie algebra over a field of characteristic zero such that its second cohomology with coefficients in any finite-dimensional module vanishes, is one of the following:*

- (i) *an one-dimensional algebra;*
- (ii) *a semisimple algebra;*
- (iii) *the direct sum of a semisimple algebra and an one-dimensional algebra.*

In the first section of this note we accumulate the needed results from the literature, and in the second section we provide the proof.

Our notations are standard. If L is a Lie algebra, $Rad(L)$ denotes the solvable radical of L . If V is an L -module, $V^L = \{v \in V \mid xv = 0 \text{ for any } x \in L\}$ is a submodule of L -invariants. Throughout the note, the ground field K assumed to be of characteristic zero and all algebras and modules assumed to be finite-dimensional. When being considered as a module over a Lie algebra, K understood as a trivial module.

1. Needed results

Proposition 1.1. (Dixmier). *Let L be a Lie algebra, V be an L -module, and I be an ideal of L of codimension 1. Then for any $n \in \mathbb{N}$ and $x \in L \setminus I$, $H^n(L, V) \simeq H^n(I, V)^x \oplus H^{n-1}(I, V)^x$.*

Proof. This was proved in an equivalent form of a certain long exact sequence in [2, Proposition 1] and stated without proof (in a more general situation when I is a subalgebra of codimension 1) in [4, Proposition 4]. In fact, this is an easy consequence of the Hochschild–Serre spectral sequence.

Let $L = I \oplus Kx$ for some $x \in L$, $x \notin I$ and consider the Hochschild–Serre spectral sequence abutting to $H^*(L, V)$ relative to the ideal I . As $E_2^{pq} = H^p(L/I, H^q(I, V)) \simeq H^p(Kx, H^q(I, V))$, it is nonzero only for $p = 0, 1$, in which cases $E_2^{0q} = H^q(I, V)^x$ and $E_2^{1q} \simeq H^q(I, M)/xH^q(I, M) \simeq H^q(I, M)^x$. Hence the spectral sequence stabilizes at E_2 , $H^n(L, V) \simeq E_\infty^{0n} \oplus E_\infty^{1, n-1} = E_2^{0n} \oplus E_2^{1, n-1}$, and the result follows. ■

Proposition 1.2. (Dixmier). *If L is a nilpotent Lie algebra of dimension > 1 , then $H^2(L, K) \neq 0$.*

Proof. In [2, Théorème 2], a much more general result is stated: $\dim H^n(L, V) \geq 2$ for any $0 < n \leq \dim L$ and any finite-dimensional L -module V containing K . It is proved by repetitive applications of Proposition 1.1. ■

And last, we will need the following simple result which implicitly contained already in the foundational paper [8] and which is explicitly proved, for example, in [10, Lemma 1]:

Proposition 1.3. (Hochschild – Serre). *Let L be a Lie algebra represented as the semidirect sum $L = P \oplus I$ of a subalgebra P and an ideal I , and V be an L -module. Then the Hochschild–Serre spectral sequence abutting to $H^*(L, V)$ relative to the ideal I , stabilizes at the E_2 term.*

2. Proof of Theorem 0.2

Finite-dimensional Lie algebras with the property that their second cohomology with coefficients in any finite-dimensional module vanishes, will be called *2-trivial*.

Lemma 2.1. *Let L be a 2-trivial Lie algebra represented as the semidirect sum $L = S \oplus I$ of a subalgebra S and an ideal I . Then:*

- (i) S is 2-trivial;
- (ii) $(H^2(I, K) \otimes V)^S = 0$ for any S -module V .

Proof. Let V be an L -module and consider the Hochschild–Serre spectral sequence abutting to $H^*(L, V)$ relative to I . By Proposition 1.3, it stabilizes at E_2 , hence all E_2 terms vanish. We have $E_2^{20} = H^2(L/I, H^0(I, V)) \simeq H^2(S, V^I)$ and $E_2^{02} = H^0(L/I, H^2(I, V)) \simeq H^2(I, V)^S$.

Choose V as follows: let V be an arbitrary S -module, and I acts on V trivially. Then $E_2^{20} \simeq H^2(S, V)$, $E_2^{02} \simeq (H^2(I, K) \otimes V)^S$, and they vanish for any S -module V , what proves (i) and (ii) respectively. ■

In particular case when the semidirect sum reduces to the direct sum, (i) shows that the direct summand of a 2-trivial Lie algebra is 2-trivial. The last assertion may be proved also directly by using the Künneth formula, without appealing to Proposition 1.3.

Lemma 2.2. *An ideal of codimension 1 in a 2-trivial Lie algebra is a direct summand.*

Proof. Let L be a 2-trivial Lie algebra and I be an ideal of L of codimension 1. Write $L = I \oplus Kx$ for some $x \in L$. Evidently adx is an x -invariant 1-cocycle in $Z^1(I, I)$. As by Proposition 1.1, $H^1(I, I)^x$ embeds into $H^2(L, I) = 0$, this cocycle is a coboundary, i.e. there is $z \in I$ such that $[y, x] = [y, z]$ for any $y \in I$. Replacing x by $x' = x - z$, we get a direct sum decomposition $L = I \oplus Kx'$, $[I, x'] = 0$. ■

Corollary 2.3. *A 2-dimensional Lie algebra is not 2-trivial.*

Proof. By Lemma 2.2, a 2-trivial 2-dimensional Lie algebra is abelian. But for any abelian Lie algebra L and any $n \in \mathbb{N}$, $H^n(L, K) = C^n(L, K)$, and hence any abelian Lie algebra of dimension > 1 is not 2-trivial. ■

Lemma 2.4. *Let L be a 2-trivial Lie algebra. Then $H^2(\text{Rad}(L), K) = 0$.*

Proof. The assertion is obvious in the case when $L = \text{Rad}(L)$ is solvable, so suppose L is not solvable. Let $L = S \oplus \text{Rad}(L)$ be a Levi–Malcev decomposition, where S is a semisimple Malcev subalgebra.

By Lemma 2.1(ii), $(H^2(\text{Rad}(L), K) \otimes V)^S = 0$ for any S -module V . Assume $H^2(\text{Rad}(L), K) \neq 0$ and take $V = H^2(\text{Rad}(L), K)^*$. There is a canonical surjection of S -modules $H^2(\text{Rad}(L), K) \otimes H^2(\text{Rad}(L), K)^* \rightarrow K$. Since any extension of S -modules splits, K contained in $H^2(\text{Rad}(L), K) \otimes H^2(\text{Rad}(L), K)^*$, hence $(H^2(\text{Rad}(L), K) \otimes H^2(\text{Rad}(L), K)^*)^S \neq 0$, a contradiction. ■

Now we are ready to prove Theorem 0.2.

Let L be a 2-trivial Lie algebra. We shall reason by induction on the dimension of L .

If $[L, L] = L$, then $\text{Rad}(L)$ is nilpotent (in fact, this is the consequence of the Levi Theorem which in turn is the consequence of the Second Whitehead Lemma; see [9], Chap. III, §9, Corollary 2). Note that $\text{Rad}(L)$ cannot be one-dimensional, as then $\text{Rad}(L)$ is one-dimensional representation of a semisimple Lie algebra, and hence is a trivial representation, what contradicts the condition $[L, L] = L$. If $\dim \text{Rad}(L) > 1$, then by Proposition 1.2, $H^2(\text{Rad}(L), K) \neq 0$, while by Lemma 2.4, $H^2(\text{Rad}(L), K) = 0$, a contradiction. Hence $\text{Rad}(L) = 0$, i.e. L is semisimple.

Let $[L, L] \neq L$. As any subspace of L containing $[L, L]$ is an ideal of L , L contains an ideal I of codimension 1. By Lemma 2.2, L is the direct sum of I and an one-dimensional algebra, and by Lemma 2.1(i), I is 2-trivial. By induction assumption, I is either one-dimensional, or semisimple, or the direct sum of a semisimple algebra and an one-dimensional algebra. In the first case L is abelian 2-dimensional, a contradiction (Corollary 2.3). In the third case L is the direct sum of a semisimple algebra and a 2-dimensional abelian algebra. By Lemma 2.1(i), both direct summands are 2-trivial, the same contradiction again. Hence the only possible case is when L is the direct sum of a semisimple algebra and an one-dimensional algebra, what concludes the proof.

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