QUESTIONS ON LIE ALGEBRAS OF COHOMOLOGICAL DIMENSION 1

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A Lie algebra L is said to be of *cohomological dimension 1*, if $H^2(L, M) = 0$ for any L-module M. Due to the standard interpretation of the second cohomology group, this is equivalent to the condition that each exact sequence

$$0 \to M \to G \to L \to 0$$

splits. Of course, the cohomological dimension may be defined via standard devices of homological algebra – e.g. as the minimal length of the projective resolution. In particular, the latter definition shows, that for Lie algebras of cohomological dimension 1 all higher cohomology groups also vanish.

The similar notion may be defined for other classes of algebraic systems, such as groups and associative algebras.

Note that we consider the category of *all* L-modules, including infinite-dimensional ones. If we restrict ourselves with, say, finite-dimensional Lie algebras and the category of finite-dimensional modules, the whole subject, both in results and methods employed, becomes quite different (one could mention the classical Whitehead Lemmata for semisimple Lie algebras in characteristic zero and Dzhumadil'daev–Farnsteiner–Strade non-vanishing result in characteristic p). We will not touch this subject here.

Obviously, a free Lie algebra, due to its universal property, has cohomological dimension 1. The question is whether the converse is true. This was asked several times, among them by Bourbaki [B, footnote to Ex. II.2.9] and in [MZ, Problem 28.11].

In the late 60s, Stallings [St1], [St2] (for the case of finitely-generated groups) and Swan [Sw] (for the general case) proved that a group G of cohomological dimension 1 is free (where cohomology understood in the category of $\mathbb{Z}G$-modules). Their initial reasonings were based on the notion of ends of topological spaces and contained a good deal of topology. A streamlined and purely algebraic proof may be found in [Co], a more recent nice surveys of these and related results and ideas may be found in [Ca, §6] and [Ea], and a guide through the original Stallings–Swan proof may be found in [CZ, §6.2] and [St3, §4-5] (the latter one, due to Stallings himself, also contains a very clear introduction to the theory of ends). Ends in different topological and algebraic settings are discussed also in [HR] (without any connection to Stallings–Swan theorem), and probably the study of the latter book may reveal new approaches. Dunwoody [D] extended the Stallings-Swan theorem to the category of RG-modules for arbitrary coefficients ring R.

The question was also considered for different classes of semigroups (see [N] and references therein. The latter paper contains also amazingly simple reasonings and it is probably worth to explore whether they are applicable in Lie-algebraic case).

Now let us turn to Lie algebras.

Shapiro's lemma about cohomology of coinduced modules implies that each subalgebra of a Lie algebra of cohomological dimension 1 is again of cohomological dimension 1.

It is relatively easy to see that if a Lie algebra L can be represented as $\mathcal{L}/I$ where $\mathcal{L}$ is a free Lie algebra, I an ideal of $\mathcal{L}$ contained in $[\mathcal{L}, \mathcal{L}]$, and L has cohomological dimension 1

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(in fact, it is sufficient to require just $H^2(L,K) = 0$), then L is free (see, for example, [B, Ex. II.2.8]).

The only finite-dimensional Lie algebra of cohomological dimension 1 is one-dimensional. This is almost immediate: say, the general case is reduced to the case of algebraically closed field by passing to the algebraic closure, and the latter case is treated by induction on the dimension of an algebra. Consequently, all finite-dimensional subalgebras of a Lie algebra of cohomological dimension 1 are one-dimensional.

In 1983 Feldman [F] proved that a 2-generated Lie algebra of cohomological dimension 1 is free. Consequently, each 2-generated subalgebra of a Lie algebra of cohomological dimension 1 is free.

In the late 1980s, G.P. Kukin and M.A. Shevelin tried to attack the problem in the following way (not published, our account is based on [G]). First, they proved the following: Let I be an ideal and B be a subalgebra of a free nonassociative algebra, and $B \cap I \subseteq I^2$. Then B can be decomposed into a free product of subalgebras: $B = B_1 * B_2$ such that $B_1 \cap I = 0$ and $B_2 \subseteq I^2$. The proof uses sophisticated combinatorial manipulations in a free nonassociative algebra. Then, they tried to derive from this the analogous result in the category of Lie algebras, and from the latter the freeness of Lie algebras of cohomological dimension 1 would follow. Unfortunately, the analogous statement for Lie algebras appeared to be wrong.

In 1994, Mikhalev, Umirbaev and Zolotykh ([MUZ1], [MUZ2], and [MZ, pp. 192–195]) constructed an example of a non-free Lie algebra of cohomological dimension 1 over a field of characteristic $p > 2$. It is an algebra generated by 3 elements x, y, z with a single defining relation

$$x + [y,z] + (ad\, x)^p(z) = 0.$$

It enjoys other interesting properties, particularly, its universal enveloping algebra is the 2-generated free associative algebra. Note that it is not a p-algebra.

Further examples of such algebras could be obtained via the free product construction. Indeed, since

$$H^\bullet(L_1 * L_2, M) \simeq H^\bullet(L_1, M) \oplus H^\bullet(L_2, M)$$

(which is just a particular case of the Mayer-Vietoris long exact sequence), the free product of Lie algebras of cohomological dimension 1 is of cohomological dimension 1 too. Thus, taking the free product of Mikhalev–Umirbaev–Zolotykh example with itself, or with a free Lie algebra, we will get non-free Lie algebras of cohomological dimension 1.

Seregin [Se] examined some properties of Lie algebras of cohomological dimension 1, particularly, gave necessary and sufficient conditions for an algebra with a single defining relation to have a cohomological dimension 1, in terms of some equations in a free associative algebra.

An interesting approach to construction of Lie algebras of small cohomological dimension was outlined by Shevelin in [Sh1]. Using ideas of the well-known composition lemma (also known as diamond lemma), he easily constructed a Lie algebra having a free 3-term resolution of a trivial module (and hence, cohomological dimension ≤ 3). It may be that one can twist arguments there to obtain another examples of non-free Lie algebras of cohomological dimension 1.

One may also consider a notion of cohomological dimension in the category of Lie p-algebras (with restricted cohomology). Some initial remarks in this setting were done by Shevelin in [Sh2]. As it happens with restricted cohomology, the situation is quite different from the ordinary cohomology – for example, finite-dimensional 1-generated abelian Lie p-algebra has infinite cohomological dimension.

(Hochschild) cohomological dimension in the class of associative algebras behaves quite differently than in Lie-algebraic and group cases (see, for example, [CQ] where associative algebras of cohomological dimension 1 are called *quasi-free* or *formally smooth*). For example, there are finite-dimensional algebras belonging to this class (in fact, in the class of finite-dimensional associative algebras, cohomological dimension 1 has been known for a long time to be equivalent to separability, see, for example, [H] and [Ei]), the situation depends on the ground field, etc. Though one may prove quite a few statements about such algebras, their full descriptions seems to be out of reach.

Finally, one may consider *homological* dimension, instead of cohomological one. Groups of homological dimension 1 were investigated in [KLL]. Surprisingly (may be naively), the picture is quite different from the cohomological counterpart. The analogous question for Lie algebras was, to our knowledge, never considered.

Questions.

(1) Is it true that over a field of characteristic zero, a Lie algebra of cohomological dimension 1 is free?
(2) Is it true that over a field of characteristic p, a Lie p-algebra of restricted cohomological dimension 1 is free?
(3) Study cohomological dimension of graded Lie algebras in the category of graded modules, and, particularly, graded Lie algebras of (graded) cohomological dimension 1.
(4) Study homological dimension in Lie algebras, and, particularly, Lie algebras of homological dimension 1.
(5) Study Lie algebras of cohomological dimension 1 inside nontrivial varieties of Lie algebras (this is probably too vague).

We stress that in question (2), cohomological dimension is understood in the sense of restricted cohomology, not ordinary one (in that sense, Conjecture 2 in [MUZ2] could be misleading). In fact, it is easy to see that a Lie p-algebra of (ordinary) cohomological dimension 1 is necessarily 1-dimensional. For, consider a subalgebra $\langle x \rangle_p$ generated by p-powers of a single element $x \in L$. $\langle x \rangle_p$ is abelian and of cohomological dimension 1, hence is 1-dimensional. Consequently, for any $x \in L$ there is $k \in K$ such that $x^p = kx$. Obviously, every subalgebra of L satisfies the same property and hence is a p-algebra. Now take a 2-generated subalgebra of L. By the above-mentioned Feldman's result, it is free. But a free non-one-dimensional Lie algebra obviously could not be a p-algebra, a contradiction.

References

[B] N. Bourbaki, *Groupes et Algèbres de Lie*, Chapitres 2-3, Hermann, 1972.

[Ca] J.F. Carlson, *The cohomology of groups*, Handbook of Algebra, Vol. 1, (ed. M. Hazewinkel), Elsevier, 2000, 581–610.

[Co] D.E. Cohen, *Groups of cohomological dimension one*, Lect. Notes Math. **245** (1972).

[CZ] D.J. Collins and H. Zieschang, *Combinatorial group theory and fundamental groups*, Combinatorial Group Theory and Applications to Geometry (ed. A.I. Kostrikin and I.R. Shafarevich), Springer, 1998, 1–166.

[CQ] J. Cuntz and D. Quillen, *Algebra extensions and nonsingularity*, J. Amer. Math. Soc. **8** (1995), 251–289.

[D] M.J. Dunwoody, *Accessibility and groups of cohomological dimension one*, Proc. London Math. Soc. **38** (1979), 193–215.

[Ea] V.R. Easson, *Groups of cohomological dimension less than or equal to 2*, Cambridge Univ. Part III course essay, May 2001, 33 pp.

[Ei] S. Eilenberg, *Algebras of cohomologically finite dimension*, Comm. Math. Helv. **28** (1954), 310–319.

[F] G.L. Feldman, *Ends of Lie algebras*, Russ. Math. Surv. **38** (1983), No.1, 182–184.

[G] O. Gatelyuk, private letter, March 7, 1992.

[H] G. Hochschild, *On the cohomology groups of an associative algebra*, Ann. Math. **46** (1945), 58–67.

[HR] B. Hughes and A. Ranicki, *Ends of Complexes*, Cambridge Univ. Press, 1996; `http://www.maths.ed.ac.uk/~aar/books/` .

[KLL] P. Kropholler, P. Linnell and W. Lück, *Groups of small homological dimension and the Atiyah conjecture*, `arXiv:math/0401312`.

[MUZ1] A.A. Mikhalev, U.U. Umirbaev, and A.A. Zolotykh, *An example of a non-free Lie algebra of cohomological dimension* 1, Russ. Math. Surv. **49** (1994), No.1, 254.

[MUZ2] ——— , ——— , and ——— , *A Lie algebra with cohomological dimension one over a field of prime characteristic is not necessarily free*, First International Tainan–Moscow Algebra Workshop, de Gruyter, 1996, 257–264.

[MZ] ——— and A.A. Zolotykh, *Combinatorial Aspects of Lie Superalgebras*, CRC Press, 1995.

[N] B.V. Novikov, *Semigroups of cohomological dimension one*, J. Algebra **204** (1998), 386–393.

[Se] A.V. Seregin, *On some properties of Lie algebras of cohomological dimension one*, Fund. Prikl. Mat. **4** (1998), No.2, 779–783 (in Russian).

[Sh1] M.A. Shevelin, *Application of method of compositions to homological algebra*, Comm. Omsk Univ., 1998, Vyp. 1, 17–19 (in Russian).

[Sh2] ——— , *Solvable Lie p-algebras without a torsion*, Matem. Struktury i Modelirovanie (Omsk), 1998, Vyp. 1, 37–47 (in Russian).

[St1] J.R. Stallings, *Groups of dimension 1 are locally free*, Bull. Amer. Math. Soc. **74** (1968), 361–364.

[St2] ——— , *On torsion-free groups with infinitely many ends*, Ann. Math. **88** (1968), No.2, 312–334.

[St3] ——— , *Group theory and three-dimensional manifolds*, Yale Mathematical Monographs, 1971.

[Sw] R.G. Swan, *Groups of cohomological dimension one*, J. Algebra **12** (1969), 585–610.

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